

Time: 2 hours

Total: 140 points

Part I: (40 points) Consider Lebesgue measure on the real line  $\mathbb{R}$ , and let

$$f_n : \mathbb{R} \rightarrow \mathbb{R}, \quad n = 1, 2, 3,$$

be a sequence of measurable functions. State whether each of the following is TRUE or FALSE. If it is true, give a brief reason; if false, give a counterexample.

- $f_n \rightarrow 0$  in  $L^3(\mathbb{R})$  and  $\|f_n\|_1 \leq 10 \forall n \implies f_n \rightarrow 0$  in  $L^2(\mathbb{R})$ .
- $f_n \rightarrow 0$  in  $L^1(\mathbb{R}) \implies f_n \rightarrow 0$  a.e.
- $f_n \rightarrow 0$  uniformly and  $f_n \rightarrow 0$  in  $L^1(\mathbb{R}) \implies f_n \rightarrow 0$  in  $L^2(\mathbb{R})$ .
- $f_n \rightarrow 0$  in  $L^2(\mathbb{R}) \implies f_n \rightarrow 0$  weak\* in  $\mathcal{D}'(\mathbb{R})$ .

Part II: Do 4 of the following 5 problems. (If you attempt all 5 problems, clearly indicate which problems you want graded.) Each problem is worth 25 points.

- Let  $f : [0, 1] \rightarrow \mathbb{R}$  be of bounded variation. Prove that  $f$  is Borel measurable.
  - Give an example of a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$  that is not of bounded variation.
- Let  $\{f_n\}$  be a sequence of pointwise-bounded, real-valued, continuous functions on  $\mathbb{R}^3$ . Prove that  $\{f_n\}$  is uniformly bounded on some open ball  $B(x_0, r) = \{x \in \mathbb{R}^3 : \|x - x_0\| < r\}$  ( $r > 0$ ).

3. Let  $f_n : X \rightarrow \mathbb{C}$  be a sequence of measurable functions on a measure space  $X$  such that

$$\sum_{n=1}^{\infty} \|f_n\|_2 < +\infty$$

Prove that  $f_n(x) \rightarrow 0$  for almost all  $x \in X$ .

4. Describe all the complex Borel measures  $\mu$  on  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  with  $\widehat{\mu}(n+5) = \widehat{\mu}(n)$  for all  $n \in \mathbb{Z}$ . Justify your answer.

5. Let  $X$  be the vector space of complex-valued Borel measures on  $\mathbb{R}$ . Prove that

$$\|\mu\| = |\mu|(\mathbb{R}) \quad (\mu \in X)$$

is a norm on  $X$  that makes  $X$  into a Banach space.