ALGEBRA CORE QUALIFYING EXAM. SEPTEMBER 25, 2000.

Directions: Solve five problems from the following list of seven and clearly indicate which problems you chose as only those will be graded. Show all your work.

In general, it is permissible to use earlier parts of a problem in order to solve a later part even if you have not solved the earlier parts.

1. Let G be a non–abelian group of order p^3, where p is a prime number. Let $Z(G)$ be its center and G' its commutator subgroup.
(a) Show that $Z(G) = G'$ and this is the unique normal subgroup of G of order p.
(b) What is the exact number of distinct conjugacy classes of G ?

2. (a) Let p be a prime number. Show that $f(X) = X^p - pX - 1$ is irreducible in $\mathbb{Q}[X]$. (Hint: use Eisenstein's criterion of irreducibility for the image of $f(X)$ via a ring automorphism of $\mathbb{Q}[X]$.)
(b) Let R be the ring $\mathbb{Z}[X]/(X^4 - 3X^2 - X)$, where $(X^4 - 3X^2 - X)$ is the ideal generated by $X^4 - 3X^2 - X$ in $\mathbb{Z}[X]$. Find all the prime ideals of R containing $\hat{3}$ (the image of $3 \in \mathbb{Z}[X]$ via the canonical surjection $\mathbb{Z}[X] \longrightarrow R$.)

3. Let K/k be a finite, separable field extension of degree n. Let

$$\rho, \quad \rho' : K \longrightarrow M_n(k)$$

be two morphisms of k–algebras, where $M_n(k)$ is the ring of $n \times n$ matrices with entries in k. Show that there exists an invertible matrix A in $M_n(k)$ such that

$$\rho'(x) = A \cdot \rho(x) \cdot A^{-1}, \text{ for all } x \in K .$$

4. Let G be a finite group. Show that there exists a Galois field extension K/k whose Galois group is isomorphic to G.

5. Let k be a field of characteristic p, where p is a prime number. Let X and T be two (algebraically) independent variables over k.
(a) Show that the degree of the field extension $k(X,T)/k(X^p,T^p)$ is p^2.
(b) Show that there exist infinitely many distinct fields F such that

$$k(X^p,T^p) \subseteq F \subseteq k(X,T) .$$

(see over, please)

6. Let R be the ring $\mathbb{Q}[X]/\left(X^7-1\right)$, where $\left(X^7-1\right)$ is the ideal generated by X^7-1 in $\mathbb{Q}[X]$. Give an example of a finitely generated projective R–module which is not R–free. (We remind you that an R-module is called projective if it is a direct summand of a free R–module.)

7. Let D_{10} be the dihedral group of order 10, given by the usual generators and relations

$$D_{10} = \left\langle r,\, s \,|\, r^5 = 1 = s^2, rs = sr^{-1} \right\rangle$$

(1) Compute the conjugacy classes of D_{10}.
(2) Compute the commutator subgroup D_{10}' of D_{10}.
(3) Show that D_{10}/D_{10}' is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and conclude that D_{10} has precisely two distinct characters of degree 1.
(4) Write down the complete character table of D_{10}.