Schur Weyl duality:

Expository talk for large N seminar

Steve Zelditch
Department of Mathematics
Johns Hopkins University

CAVEAT LECTOR: Everything has been copied from one or another reference cited below, but mistakes may have crept in. Please inform me of ones you find.

Topics in Schur-Weyl duality

These lectures concern the Schur-Weyl duality between irreps of the symmetric groups S_n and of $U(N)$ via partitions/Young diagrams.

- Young diagrams and irreps of $U(N)$ and $GL(N,\mathbb{C})$. Tensor reps $V^{\otimes k}$, Conjugate reps $V^{*\otimes k}$ and products $(\bigotimes^k V) \otimes (\bigotimes^\ell V^*)$.

- Young diagrams and irreps of $SU(N)$ and $SL(N,\mathbb{C})$.

- Schur polynomials.

- Double commutant theorem and Schur-Weyl duality for $V^{\otimes k}$;

- Dimension and character relations of corresponding $S_k - GL(N,\mathbb{C})$ irreps.

References

- H. Weyl, *Group Theory and Quantum Mechanics*. Most of the physics papers use the notation of this book.

- H. Weyl, *The Classical Groups*.

- R. Goodman and N. R. Wallach, *Representations and Invariants of the Classical Groups*. Has a modern presentation of Schur-Weyl duality for tensor reps of $GL(N,\mathbb{C})$, but does not discuss Schur polynomials, full rep theory of $GL(N,\mathbb{C})$, or dimension or character relations.

- I. G. Macdonald, Symmetric Functions and Hall Polynomials. Discussion of Schur polynomials.

- W. Fulton and J. Harris, *Representation Theory*: discusses all topics with examples.

- D.P. Zhelobenko, *Compact Lie groups and their representations*

Basic idea

The representation theory of the symmetric group S_k and of the general linear group $GL(N,\mathbb{C})$ come together in the study of the representation of $S_k \times GL(N,\mathbb{C})$ on $(\mathbb{C}^N)^{\otimes k}$, i.e. the kth tensor power of the fundamental representation of $GL(N,\mathbb{C})$. The key results are:

- The Schur duality pairing between irreps of S_k and irreps of $GL(N,\mathbb{C})$;

- The Frobenius character formula for S_k characters;

- Construction of Young symmetrizers = projection operators onto $GL(N,\mathbb{C})$-invariant subspaces.

Basics of tensor reps

All unitary irreducible reps of $SU(N)$ or $SL(N,\mathbb{C})$ are obtained from reducing tensor powers $V^{\otimes k}$ of the fundamental rep.

For $U(N)$ or $GL(N,\mathbb{C})$ we also need the determinant. See §15. 5 of Fulton-Harris. Every irrep of $GL(N,\mathbb{C})$ is a tensor product of a power of det with a tensor rep. We will discuss this further below. In the physics it is put as follows: all irreps occur of $GL(N,\mathbb{C})$ or $U(N)$ occur in $(\bigotimes^k V) \otimes (\bigotimes^\ell V^*)$ for some (k,ℓ).

Irreps of $GL(N,\mathbb{C})$

Let $H_N \subset GL(N,\mathbb{C})$ denote the subgroup of diagonal matrices, and let N_N denote the subgroup of upper-triangular unipotent matrices. The characters of H_N are parametrized by $\lambda = [\lambda_1, \ldots, \lambda_N] \in \mathbb{Z}^N$:

$$h \to h^\lambda = x_1^{\lambda_1} \cdots x_N^{\lambda_N}.$$

Weights of a rep ρ of $GL(N,\mathbb{C})$ = characters of H_N which occur in $\rho|_{H_N}$.

Dominant weights $\lambda \in \mathbb{N}^N_{++} := \lambda \in \mathbb{Z}^N,\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Those which occur in tensor reps have $\lambda_N \geq 0$.

$(\mathbb{C}^N)^{\otimes k}$

$(\mathbb{C}^N)^{\otimes k}$ is spanned by linear combinations of decomposable elements $v_1 \otimes \cdots \otimes v_k$ where $v_j \in \mathbb{C}^N$.

We define the mutually commuting representations:

- ρ_k of $GL(N, \mathbb{C})$ by

  $$\rho_k(g) v_1 \otimes \cdots \otimes v_k = g \cdot v_1 \otimes g \cdot v_2 \otimes \cdots \otimes g \cdot v_k.$$

- σ_k of S_k by:

  $$\sigma_k(s) v_1 \otimes \cdots \otimes v_k = v_{s^{-1}(1)} \otimes v_{s^{-1}(2)} \otimes \cdots \otimes v_{s^{-1}(k)}.$$

  Thus, $\sigma_k(s)$ moves the vector in the ith position to the position $s(i)$, but doesn't change their values.

Schur-Weyl duality (d'après Goodman-Wallach)

Theorem **1** *Under the action of* $S_k \times GL(N, \mathbb{C})$*, we have the decomposition,*

$$\bigotimes^{k} (\mathbb{C}^N) \simeq \bigoplus_{\lambda \in Par(k,N)} F_N^\lambda \otimes G^\lambda,$$

where:

- $Par(k, N)$ *is the set of partitions of* k *with* $\leq N$ *parts;*
- F_N^λ *is the irrep of* $GL(N, \mathbb{C})$ *with highest weight* λ*;*
- G^λ *is the irrep of* S_k *corresponding to* λ*.*

Remarks

As we will see below, the group algebras of S_k and $GL(N, \mathbb{C})$ are each other's commutants in $End((\mathbb{C}^N)^{\otimes k})$. By an abstract result, the double commutant theorem, one has

$$\bigotimes^{k}(\mathbb{C}^N) \simeq \bigoplus_i V_i \otimes W_i,$$

where V_i are mutually inequivalent reps of $GL(N, \mathbb{C})$ and where W_j are m.i.r. of S_k. What the Schur-Weyl duality theorem does is:

- Determine which reps of $GL(N, \mathbb{C})$, resp. S_k occur;

- The explicit pairing.

Some notation

Let $e_1, \ldots, e_N$ denote the standard basis of $\mathbb{C}^N$. The standard basis for $(\mathbb{C}^N)^{\otimes k}$ is then:

$$e_I = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k},$$

$$\text{where} \quad I = (i_1, \ldots, i_k) \subset \{1, \ldots, N\}.$$

Note: the indices need not be distinct, e.g. one could have $I = (1, \ldots, 1)$. We can view I as a map

$$I : \{1, \ldots, k\} \to \{1, \ldots, N\}.$$

We denote the set of such maps by $\{1, \ldots, N\}^k$.

S_k acts on $\{1, \ldots, N\}^k$, by

$$I = \{i_1, \ldots, i_k\} \implies s \cdot I = \{i_{s(1)}, \ldots, i_{s(k)}\},$$

i.e. it permuates the domain of I.

Tensors and linear transformations

A tensor $a \in V^{\otimes k}$ may be written in terms of the basis e_I as

$$a = \sum_{I \in \{1,\ldots,N\}^k} a_I e_I.$$

Weyl identifies a with a function

$$a : \{1, \ldots, N\}^k \to \mathbb{C}, \;\; a(I) = a_I.$$

So the action of S_k on tensors is induced from its action on $\{1, \ldots, N\}^k$:

$$s \cdot a(I) = a(s \cdot I).$$

Let $T \in End(\mathbb{C}^N)^{\otimes k})$ be a linear transformation. The matrix of T relative to this basis is $a_{I,J}$ where $a_{I,J} = \langle Te_I, e_J \rangle$, i.e.

$$Te_I = \sum_J a_{I,J} e_J.$$

Irreps in $\otimes^k(\mathbb{C}^N)$

Write $Par(k)$ for partitions of $k \in \mathbb{N}$ and $Par(k, N)$ for partitions with N parts.

Proposition **2** *The weights of H_N which occur in $\bigotimes^k(\mathbb{C}^N)$ are all $\lambda \in \mathbb{N}^N$ with $\lambda_1+\cdots+\lambda_N = k$, i.e. $\lambda \in Par(k, N)$. The corresponding weight space is:*

$$(\bigotimes^k(\mathbb{C}^N))(\lambda) = \mathit{Span}\,\{e_I : \mu_I = \lambda\}.$$

Here, $\mu_I = \lambda$ means: $\lambda_i =$ the multiplicity of i in I.

Therefore the possible irreps of $GL(N, \mathbb{C})$ which occur are those F_N^λ with highest weight $\lambda \in Par(k, N)$.

Proof

Write $V = \mathbb{C}^N$. Then the weight space of $V^{\otimes k}$ of weight $\lambda \in \mathbf{N}^N$ is:

$$V(\lambda) = \{v \in V^{\otimes k} : \rho_k(t)v = t_1^{\lambda_1} \cdots t_N^{\lambda_N} v\}.$$

Given $I : \{1, \dots, k\} \to \{1, \dots, N\}$ let

$\mu_j = \#\{p : i_p = j\} = \#I^{-1}(j), \quad \mu_I = (\mu_1, \dots, \mu_N).$

Then: $\rho_k(t)e_I = t^{\mu_I} e_I$. So the weights $= \{\mu_I\}$ and weight vectors $= e_I$. Since $|\mu_I| = k$, $V^{\otimes k}(\lambda) = \{0\}$ unless $|\lambda| = k$ and in that case,

$$V^{\otimes k}(\lambda) = Span\{e_I : \mu_I = \lambda\}.$$

Conjugacy classes of S_k

A conjugacy class C in S_k is determined by the cycle decomposition of elements in the class. Write

$$C = C(1^{i_1} 2^{i_2} \cdots k^{i_k})$$

for the class of elements with i_1 cycles of length 1, i_2 cycles of length 2 and so on. E.g. for $k = 4$, $(1)(2)(3)(4) \in C(1^4)$ and $(1)(2)(34) \in C(1^2 2^1)$. The cycle lengths satisfy

$$1 i_1 + 2 i_2 + \cdots + k i_k = k.$$

Clearly, the cycle lengths, enumerated with multiplicity, are in $1-1$ correspondence with partitions of k, namely a partition of k corresponds to a cycle structure $\cdots q^{i_q} \cdots$ if it has i_q parts of size q.

Identifying the multiplicity spaces

We claim:

$$\bigotimes^{k}(\mathbb{C}^N) \simeq \bigoplus_{\lambda \in Par(k,N)} F_N^\lambda \otimes G_{N,k}^\lambda,$$

where

$$G_{N,k}^\lambda = (\bigotimes^{k} \mathbb{C}^N)^{N_N}(\lambda),$$

i.e. the space of tensors of weight λ fixed by the upper triangular unipotent subgroup.

This follows from the theorem of the highest weight: namely, the multiplicity space for F_N^λ $\simeq$ the space of highest weight vectors in this isotypic summand. Highest weight = annihilated by $\mathbf{n}$ $\iff$ fixed by $e^{\mathbf{n}}$. Here, $\mathbf{n}$ = strictly upper triangular matrices.

Identifying the multiplicity spaces

Next, we claim

Proposition **3** *Let* $\lambda \in Par(k,p)$. *Then for* $N \geq p$,

$$(\bigotimes^k \mathbb{C}^N)^{N_N}(\lambda) = (\bigotimes^k \mathbb{C}^p)^{N_p}(\lambda) := \sigma^\lambda.$$

Proof: The embedding $\mathbb{C}^p \subset \mathbb{C}^N$ for $p < N$ gives an embedding $(\bigotimes^k \mathbb{C}^p) \subset (\bigotimes^k \mathbb{C}^N)$. From the description of the weight space,

$$(\bigotimes^k (\mathbb{C}^N))(\lambda) = \mathrm{S}pan\,\{e_I : \mu_I = \lambda\},$$

we see that for $i > p$, then $\rho_k(E_{i,i+1})u = 0$ for any $u \in (\bigotimes^k \mathbb{C}^N)(\lambda)$. Here, $E_{i,j}$ are the usual basis matrices. But $\mathbf{n}_N$ is generated as a Lie algebra by $E_{12}, E_{23}, \ldots, E_{N-1,N}$, so the Lie groups N_p and N_N have the same fixed points in $(\bigotimes^k \mathbb{C}^N)^{N_N}(\lambda)$.

Double commutant theorem

For any vector space V and subset $S \subset End(V)$, let

$$Comm(S) = \{x \in End(V) : xs = as, \ \ \forall s \in S\}.$$

Let $V = (\mathbb{C}^N)^{\otimes k}$ and let $\mathcal{A} = \rho_k(\mathbb{C}[GL(N, \mathbb{C})]), \mathcal{B} = \sigma_k(\mathbb{C}[S_k])$ be the representations of the group algebras in $End(V)$.

Theorem **4** *We have:*

- $Comm(\mathcal{B}) = \mathcal{A}$,
- $Comm(\mathcal{A}) = \mathcal{B}$.

Sketch of proof [GW, Section 3.3]

By a general double commutant theorem, if $\mathcal{A} \subset End(V)$ is a semi-simple algebra, then $\mathcal{B} = Comm(\mathcal{A})$ is s.s. and $Comm(\mathcal{B}) = \mathcal{A}$.

In our case, $\mathcal{B} = \sigma_k(\mathbb{C}[S_k])$ is s.s. (as the group algebra of a finite group, so it suffices to show that $Comm(\mathcal{B}) = \mathcal{A}$. It is clear that $Comm(\mathcal{B}) \subset \mathcal{A}$, i.e. $\sigma_k(s)$ commutes with $\rho_k(g)$ for all s, g.

Now, $\sigma_k(s)e_I = e_{s.I}$, where

$$s \cdot \{i_1, \ldots, i_k\} = \{i_{s^{-1}(1), \ldots, s^{-1}(k)}\}.$$

Thus, $\sigma_k(s)$ changes the positions (1 to k) of the indices, not their values (1 to N).

Sketch of proof [GW, Section 3.3]

The trace form $(X, Y) = TrXY$ on $End((\mathbb{C}^N)^k)$ is non-degenerate. So $\mathcal{A} = Comm(\mathcal{B})$ comes down to:

$$(T, \mathcal{A}) = 0,\ T \in Comm(\mathcal{B}) \implies T = 0.$$

We first describe $Comm(\mathcal{B})$

$$\begin{cases} T(\sigma_k(s)e_J) = \sum_I \;\; a_{I,s\cdot J} \;\; e_I, \\ \\ \sigma_k(s)(Te_J) = \sum_I \; a_{IJ} \, e_{s\cdot I} = \sum_I \; a_{I,s^{-1}\cdot J} \, e_I. \end{cases}$$

Then $T \in Comm(\mathcal{B})$ iff

$$a_{s\cdot I, s\cdot J} = a_{IJ}, \quad \forall I, J, s.$$

Sketch of proof [GW, Section 3.3]

Also describe $\rho_k(g)$ in coordinates: If $ge_j = \sum_k g_{jk}e_k$ then the matrix of $\rho_k(g)$ equals

$$g_{I,J} = g_{i_1 j_1} \cdots g_{i_k j_k}.$$

So if $[a_{IJ}]$ is the matrix of T,

$$(T, \mathcal{A}) = 0 \iff \sum_{I,J} a_{I,J} g_{i_1 j_1} \cdots g_{i_k j_k} = 0,$$

for all $g \in GL(N, \mathbb{C})$.

The right side is a polynomial function on $GL(N, \mathbb{C})$ and extends as one to $Mat(N, \mathbb{C})$.

Conclusion of proof [GW, Section 3.3]

We then use the action of S_k, $s \cdot (I, J) = (s \cdot I, s \cdot J)$ on pairs of indices to group terms. The main observations are:

- $x_{s\gamma} = x_\gamma$, i.e. the monomials are constant on orbits;
- As noted above, the matrix elements are constant on orbits: $a_{s\cdot I, s\cdot J} = a_{IJ}$.

Hence, grouping into orbits, the equation boils down to

$$\sum_{\gamma \in \Gamma} n_\gamma a_\gamma x_\gamma = 0$$

where n_γ is the cardinality of the orbit. But x_γ are linearly independent, so $a_{I,J} = 0$ for all I, J.

Young symmetrizers

So far, to each partition of k with at most N parts ($=$ Young diagram with k boxes and $\leq N$ rows), we have associated:

- An irrep F_N^λ of $GL(N, \mathbb{C})$ with highest weight λ;
- An irrep $\sigma^\lambda, G^\lambda$ of S_k.

We now construct projection operators p_A on $V^{\otimes k}$ with range equal to irreducible subspaces of type F_N^λ corresponding to Young tableaux A.

Young tableaux

A Young tableau of shape λ is an assigment of the integers $1, \ldots, k$ to the k boxes of λ, each box receiving a different integer. Write A_{ij} for the number in the jth box of the ith row. Denote by $A(\lambda)$ the tableau formed by numbering the boxes consecutively down the columns from left to right.

Denote by $Tab(\lambda)$ the set of all Young tableaux of shape λ. S_k acts simply transitively by permuting the numbers in the boxes.

Young tableaux (cont.)

Given a tableau R with $|R| = k$, we associate the decomposable tensor

$$e_R = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k},$$

where $i_j = r$ if j occurs in the rth row of R. Thus, the numbers in the first row of A give the tensor positions in e_A corresponding to e_1, the numbers in the second row given the positions of e_2, etc. E.g. if $\lambda = [3, 2, 1, 1]$, then

$$e_{A(\lambda)} = e_1 \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_1 \otimes e_2 \otimes e_1.$$

Clearly, $\{e_A\}$ gives a basis of $V^{\otimes k}$ as A runs over $Tab(\lambda)$. Moreover S_k acts on tableau and one has $e_{s(A)} = \sigma_k(s)e_A$.

row symmetrizers
column anti-symmetrizers

A tableau with r rows partitions $\{1, \ldots, k\}$ into r disjoint subsets $R_1, \ldots, R_r$, namely the numbers in each row.

Define: $s \in S_k$ preserves the rows of A if it preserves each subset R_j. Similarly, the c columns of A form a partition of $\{1, \ldots, \}$ and one defines: s preserves the columns. Then put:

$Row(A) = \{s \in S_k :$ s preserves the rows of A;

$Col(A)\{s \in S_k :$ s preserves the columns of $A.\}$

row symmetrizers column anti-symmetrizers

Then we have:

$$\sigma_k(s)e_A = e_A \iff s \in Row(A)$$

and

$$e_A = e_B \iff B = s \cdot A$$

with $s \in Row(A)$.

Define the row symmetrizer:

$$r(A) = \sum_{r \in Row(A)} r$$

and the column anti-symmetrizer

$$c(A) = \sum_{c \in Col(A)} sign(c)c.$$

Basic properties

We have:

- $r(A)x = xr(A) = r(A)$ if $x \in Row(A)$. $c(A)x = xc(A) = sgn(x)c(A)$ if $x \in Col(A)$.

- $r(s \cdot A) = sr(A)s^{-1}$ and $c(s \cdot A) = sc(A)s^{-1}$.

- If $A \in Tab(\lambda)$ has $\leq N$ rows, then $r(A), c(A)$ preserve $(\mathbb{C}^N)^{\otimes k}(\lambda)$.

Young symmetrizer

For $A \in Tab(\lambda)$ define the Young symmetrizer corresponding to the Young tableau by

$$s(A) = c(A)r(A).$$

Theorem **5** *Let* $\lambda \in Par(k, N)$ *and let* $A \in Tab(\lambda)$*. Then the operator*

$$p_A = \frac{\dim G^\lambda}{k!} s(A) : (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes k}$$

projects onto an irrep of $GL(N, \mathbb{C})$ *of highest weight* λ*.*

Example

Let $k = 3$ and consider the partition $3 = 2 + 1$. As an element of the group algebra $\mathbb{C}[S_k]$, the Young symmetrizer is

$$s(A) = 1 + e_{(12)} - e_{(13)} - e_{(132)},$$

where we have a vector e_s for each $s \in S_k$. Indeed,

$$r_{(2,1)} = 1 + e_{(12)}, \quad c_{(21)} = 1 - e_{(13)},$$

and $s(A) = r_{(2,1)} c_{(21)} = (1 + e_{(12)})(1 - e_{(13)})$.

More on Young symmetrizers

Note that

$$Im(r(A)) = Sym^{\lambda_1}V \otimes \cdots \otimes Sym^{(\lambda_k}V,$$

while

$$In(c(A)) = \bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_2} V$$

where $\mu = \lambda^*$.

Sketch of proof

There are three main steps.

First: prove that p_A is a projection operator: $p_A^2 = p_A^* = p_A$.

Second: Use Schur duality to show:

$$\begin{aligned} p_A(\otimes^k \mathbb{C}^N) &= p_A(\oplus_{\lambda \in Par(k,N)} F_N^\lambda \otimes G_\lambda \\ &= F_N^\lambda \otimes p_A(G_\lambda). \end{aligned}$$

Finally, show: $p_A(G_\lambda) = \mathbb{C}\ c(A)e_A$. A subspace $p_A(\otimes^k \mathbb{C}^N)$ is called a Weyl module.

Weyl modules: examples

- If $\lambda = (N, 0, \cdots)$ is one row, and $A = A(\lambda)$ numbers the boxes in the row consecutively, then $s(A) = \sum_{s \in S_k} s$ and the Weyl module is $Sym^k(\mathbb{C}^N)$.

- If $\lambda = (1, 1, \cdots, 1)$ is one column, and $B = B(\lambda)$ numbers the boxes consecutively downward, then $s(B) = \sum_{s \in S_k} sign(s)s$ and the Weyl module is $\wedge^k(\mathbb{C}^N)$.

Weyl modules: examples For a less canonical example, let $k = 3$ and take the tableau to be $\lambda = [2,1]$ and $C = A(\lambda)$. Then

$$P_C = \tfrac{2}{3!}(1 - (12)(1 + (13))$$

$$= \tfrac{1}{3}(1 - (12) + (13) - (321)).$$

The Weyl module consists of tensors

$$u = u_{ijk} e_i \otimes e_j \otimes e_k,$$

with

$$u_{ijk} = \frac{1}{2}(u_{kji} - u_{jik} - u_{kij}).$$

Isotype projector version 1

Define:

$$\mathbf{P}_\lambda = (\frac{(\dim G^\lambda}{k!})^2 \sum_{A \in Tab(\lambda)} s(A).$$

Theorem **6** *Let* $\lambda \in Par(k, N)$. *Then for any subspace* $F \subset \otimes^k \mathbb{C}^N$ *invariant under* $\rho_k(GL(N, \mathbb{C}))$, $\mathbf{P}_\lambda F$ *is the isotypic subspace of* F *of type* F_N^λ *for* $GL(N, \mathbb{C})$. *If* G *is any subspace invariant under* $\sigma_k(S_k)$, *then* $\mathbf{P}_\lambda G$ *is the isotypic subspace of type* G^λ *for* S_k.

Isotype projector version 2

Background on reducing representations: Let (ρ, V) be a rep of a finite group G. For $\lambda \in \widehat{G}$, let $m_\rho(\lambda)$ be the multiplicity of λ in ρ. Let P_λ be the orthogonal projection onto the λ-isotypic component of ρ. Then:

Proposition **7** *(GW, Corollary 3.4.10). We have:*

$$P_\lambda = \frac{d_\lambda}{|G|} \sum_{g \in G} \overline{\chi_\lambda(g)}\, \rho(g).$$

More on Irreps of $GL(N,\mathbb{C})$

We can describe the Weyl modules F_N^λ more explicitly, as in Fulton-Harris, §15.5. The Weyl module $p_A(\bigotimes^k \mathbb{C}^N)$ depends up to isomorphism only on the shape of A, so one writes it $\mathbf{S}_\lambda V$ where $V = \mathbb{C}^N$.

For $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{N}^N$, define

$$A^{\mathbf{a}}V = Sym^{a_1}V \otimes Sym^{a_2}(\bigwedge^2 V) \otimes \cdots \otimes Sym^{a_N}(\bigwedge^N V).$$

Define $\Phi_{\mathbf{a}}$ to be the subrep generated by the (highest weight) vector:

$$v = (e_1)^{a_1} \cdot (e_1 \wedge e_2)^{a_2} \cdot \cdots \cdot (e_1 \wedge \cdots \wedge e_N)^{a_N}.$$

$\Phi_{\mathbf{a}}$ is an irrep;

More on Irreps of $GL(N,\mathbb{C})$

Note that a_N affects only the power of the determinant. Let $D_k = \det^k$ be the 1D irrep of $GL(N,\mathbb{C})$ and let D_{-k} be its dual. Then

$$\Phi_{(a_1,\ldots,a_N+k)} = \Phi_{(a_1,\ldots,a_N)} \otimes D_k.$$

Proposition **8** $\Phi_{\mathbf{a}} = \Psi_\lambda := \mathbf{S}_\lambda V$ *if* $\lambda = (a_1 + \cdots + a_N, a_2 + \cdots + a_N, \ldots, a_N)$.

Note that $\Psi_{\lambda+(k,k,k,\ldots,k)} = \Psi_\lambda \otimes D_k$. So one may define Ψ_λ for any dominant weight $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, not necessarily ones with $\lambda_N \geq 0$.

Proposition **9** *Every irrep of* $GL(N,\mathbb{C})$ *or* $U(N)$ *is* Φ_λ *for some dominant weight* λ.

Irreps of S_k

The Young symmetrizer $s(A)$ is a projection in the group algebra $\mathbb{C}[S_k]$ of S_k. The irreps it defines depends only on its shape λ, and we then write the Young projector as c_λ. Then the irrep of S_k corresponding to λ is the space

$$V_\lambda = \mathbb{C}[S_k]c_\lambda.$$

For instance if λ has just one row, then $c_\lambda = \sum_{s \in S_k} s$ and it is easy to see that $V_\lambda = \mathbb{C}c_\lambda$, i.e. it is the trivial rep of S_k. See Theorem 4.3 of Fulton-Harris.

χ_λ versus ϕ_C

Partitions have come up in two ways in the rep theory of S_k: they enumerate irreps, and their enumerate conjugacy classes.

There is a kind of duality under the Fourier transform: For each conjugacy class C of S_k, let ϕ_C denote its characteristic function. For each irrep λ, denote its character by χ_λ. The sets:

$$\{\phi_C\}_{C \in Conj(S_k)}, \quad \{\chi_\lambda\}_{\lambda \in \widehat{S}_k}$$

have the same cardinality but are hard to relate (see GW, p. 154: this is a kind of uncertainty principle). The key formula is:

$$\widehat{\phi}_C(\lambda) = \frac{|C|}{|S_k|}\chi_\lambda(C).$$

Characters in Weyl's notation

Let $|a^{N-1}, \cdots, a, 1|$ denote the matrix where the N rows are obtained by replacing a successively by $a_1, \ldots, a_N$. Let $D(a_1, \ldots, a_N)$ denote its determinant

$$\Pi_{i<j}(a_i - a_j).$$

Put

$$\Delta = D(\epsilon_1, \ldots, \epsilon_N).$$

For integers $h_1 > h_2 > \cdots > h_N$ define

$$\xi(h_1, \ldots, h_N) = \sum_{s \in S_N} sign(s) e(h_1\theta_{s(1)} + \cdots + h_N\theta_{s(N)}).$$

Thus,

$$\Delta = \xi(N-1, N-2, \ldots, 1, 0).$$

Weyl writes in GTQM (p. 379)

Characters in Weyl's notation

Suppose that the highest term in a character χ has exponents $f_1 \geq f_2 \geq \cdots \geq f_N$. Then

$h_1 = f_1 + (N-1),\ \ h_{N-1} = f_{N-1} + 1,\ h_N = f_N.$

One then has

$$\chi = \frac{\xi(h_1, h_2 \ldots, h_N)}{\Delta} = \frac{|\epsilon^{h_1}, \epsilon^{h_2}, \cdots, \epsilon^{h_N}|}{|\epsilon^{N-1}, \ldots, \epsilon, 1|}.$$

Symmetric functions (d'après Macdonald)

There is a close relation between:

- The graded ring $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$ of symmetric functions in infinitely many variables;

- The graded ring $R = \bigoplus_{n \geq 0} R^n$ of characters of symmetric groups. Here, R^n is the $\mathbb{Z}$-module generated by irreducible characters of S_n;

Basic symmetric functions

- Power sums: $p_r(x) = \sum_i x_i^r$;

- For a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, define

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots .$$

- Monomial symmetric functions

$$m_\lambda = \sum_s x^{s \cdot \lambda}.$$

- Complete symmetric functions

$$h_r = \sum_{\lambda : |\lambda| = r} m_\lambda.$$

  Also put

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots .$$

Precise def. of Λ:

Let $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ denote the symmetric polynomials in n variables. It is a graded ring, $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$, where Λ_n^k are those of degree k.

For $m \geq n$, define $\rho_{m,n} : \Lambda_m \to \Lambda_n$ by sending $x_{n+i} \to 0$. Then define Λ^k to be the inverse limit of Λ_n^k as $n \to \infty$: An element is a sequence (f_n) of homogeneous symmetric polynomials with $f_m(x_1, \ldots, x_n, 0, \ldots, 0) = f_n(x_1, \ldots, x_n)$.

Fact: Λ is the free $\mathbb{Z}$-module generated by the monomial symmetric functions m_λ as λ varies over all partitions.

Characteristic map

The characteristic map is the isometric isomorphism:

$$ch : R \to \Lambda_{\mathbb{C}}$$

defined as follows: for $f \in R^n$ put

$$ch(f) = \sum_{|\rho|=n} z_\rho^{-1} f_\rho p_\rho,$$

where f_ρ is the value of f on the conjugacy class C_ρ, and $z_\rho = \sum_k k^{m_k} m_k!, m_k(\lambda) = \#\{parts = k\}$. [Conjugacy classes C_ρ of S_n are indexed by partitions ρ of n: if the order of the cycles are $\rho_1 \geq \rho_2 \geq \cdots$ then $\rho(s) = (\rho_1, \rho_2, \cdots)$ is a partition of n, the cycle type.]

Characteristic map

The normalizing constants z_λ make ch an isometry w.r.t. natural inner products:

$$\begin{aligned}\langle ch(f), ch(g)\rangle_\Lambda &= \textstyle\sum_{|\rho|=n} z_\rho^{-1} f_\rho g_\rho \\ &= \langle f, g\rangle_{S_n}.\end{aligned}$$

Here,

$$\langle f, g\rangle_{S_n} = \frac{1}{|S_n|}\sum_{x\in S_n} f(x)g(x^{-1}),$$

and

$$\langle p_k, p_j\rangle_\Lambda = \delta_{jk}k.$$

If the p_k are used as coordinates, then this inner product is the Gaussian measure

$$e^{-\sum |p_k|^2/k}\prod \frac{dp_k d\bar{p}_k}{2\pi i k}.$$

Schur polynomials

Let $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ be a monomial, and let

$$A_\alpha(x) = \sum_{s \in S_N} \text{sign}(s) s \cdot x^\alpha$$

where $s \cdot x^\alpha = x_{s^{-1}(1)}^{\alpha_1} \cdots x_{s^{-1}(N)}^{\alpha_N}$. Clearly, $A_\alpha(s \cdot x) = sign(s) A_\alpha(x)$.

Clearly, $A_\alpha = 0$ unless α_i are distinct, and we assume $\alpha_1 > \alpha_2 > \cdots \alpha_N \geq 0$. We may write

$$\alpha = \lambda + \delta, \text{ where } \lambda \in Par(N),$$

and $\delta = (N-1, N-2, \ldots, 1, 0)$.

Schur polynomials (cont.)

We then observe that

$$A_{\lambda+\delta} = \mathrm{det}[x_i^{\lambda_j+N-j}]_{1\le i,j\le N}.$$

The determinant is divisible by each factor $(x_i - x_j), i < j$, hence by their product, the Vandermonde determinant:

$$A_\delta(x) = \Pi_{i<j}(x_i - x_j) = \mathrm{det}[x_i^{N-j}]_{1\le i,j\le N}.$$

Definition: The Schur (symmetric) polynomial is the ratio

$$s_\lambda = \frac{A_{\lambda+\delta}}{A_\delta}.$$

Characters of S_n

Let η_k denote the identity character of S_k. For any partition λ of S_n, put

$$\eta_\lambda = \eta_{\lambda_1} \cdot \eta_{\lambda_2} \cdots ,$$

where

$$f \cdot g = Ind_{S_m \times S_n}^{S_{m+n}} (f \times g).$$

Thus, η_λ is the character induced by the identity character of $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$.

Power sums vs. Schur (Macdonald notation)

Let χ^λ be the irreducible character of S_n associated to the partition λ.

Proposition **10** *Let* $\lambda \in Par(n)$*. Then*

$$s_\lambda = \sum_{|\rho|=n} z_\rho^{-1} \chi_\rho^\lambda p_\rho.$$

Proof: Show $ch(\chi^\lambda) = s_\lambda$. Use two determinant identities:

$$\chi^\lambda = \det[\eta_{\lambda_i - i + j}]_{i,j \le n}$$

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{i,j \le n}.$$

ch is a ring homomorphism, so the proof is concluded from the fact that $ch(\eta_\lambda) = h_\lambda$.

Power sums vs. Schur

The formula implies:

$$\langle s_\lambda, p_\rho \rangle = \chi^\lambda_\rho.$$

Thus, the transition matrix from power sums to Schur polynomials as bases of symmetric functions is the character table of S_n. This shows:

$$p_\rho = \sum_\lambda \chi^\lambda_\rho s_\lambda.$$

Character relations (Moore notations)

Define the class functions

$$\Upsilon_{\vec{k}}(U) = \Pi_{j=1}^{m}(TrU^{k_j}), \ \ k = (k_1, \ldots, k_m).$$

They are related to characters of irreducible reps by:

$$\begin{aligned} \chi_{R(Y)}(U) \ &= \textstyle\sum_{\sigma \in S_n} \frac{1}{n!} \chi_{r(Y)}(\sigma) \Upsilon_{\vec{k}(\sigma)}(U) \\ \Upsilon_{\vec{k}(\sigma)}(U) \ &= \textstyle\sum_{Y \in \mathcal{Y}_n^{(N)}} \chi_{r(Y)}(C(\vec{k})) \chi_{R(Y)}(U). \end{aligned}$$

Here we identify conjugacy classes $C(\vec{k})$ of S_n with partitions $\vec{k}$ of n, while $\vec{k}(\sigma)$ encodes the conjugacy class of an element σ, i.e. its cycle structure.

Proof of character relations

We combine two facts about the Young symmetrizer P_Y:

$$\begin{cases} P_Y V^{\otimes k} = R(Y) \otimes r(Y), \\ \\ P_Y = \frac{d_R(Y)}{k!} \sum_{\sigma \in S_k} \chi_{r(Y)}(\sigma)\sigma. \end{cases}$$

The second line restates the isotypic projector formula above (slide 32) .

We then evaluate $\chi_R(U) = Tr_{V^{\otimes k}}(UP_R)$ in two ways using each side. We claim:

$$\begin{aligned} Tr_{V^{\otimes k}}[U\sigma] &= \sum_{i_1,\ldots,i_k=1}^{N} U_{i_1 i_{\sigma(1)}} U_{i_2 i_{\sigma(2)}} \cdots U_{i_k i_{\sigma(k)}} \\ &= \Upsilon_{k(\sigma)}(U). \end{aligned}$$

Proof concluded

Indeed,

$$\begin{aligned} Tr_{V^{\otimes k}}[U\sigma] &= \sum_{I\in\{1,\ldots,N\}^k}\langle U\sigma e_I, e_I\rangle \\ &= \sum_{I\in\{1,\ldots,N\}^k}\langle U e_{\sigma(I)}, e_I\rangle \\ &= sum_{I\in\{1,\ldots,N\}^k} U_{I,\sigma(I)}, \end{aligned}$$

where we recall

$$U_{I,\sigma(I)} = U_{i_1 i_{\sigma(1)}} U_{i_2 i_{\sigma(2)}} \cdots U_{i_k i_{\sigma(k)}}.$$

Frobenius' character formula

Let $\mathbf{i} = (i_1, \ldots, i_d)$ with $\sum \alpha i_\alpha = d$ denote a cycle structure in S_d: i_1 cycles of length 1, i_2 cycles of length 2, etc. Then

$$\chi_\lambda(C_{\mathbf{i}}) = [\Delta(x)\Pi_j p_j(x)^{i_j}]|_{(\ell_1,\ldots,\ell_k)},$$

where $[f(x)]|_{(\ell_1,\ldots,\ell_k)}$ denotes the coefficient of the monomial $x_1^{\ell_1} \cdots x_k^{\ell_k}$ in $f(x)$. Also

$$\ell_1 = \lambda_1 + k - 1,\ \ \ell_2 = \lambda_2 + k - 2, \cdots, \ell_k = \lambda_k.$$

Ex: d = 5, $\lambda = (3, 2), C_{\mathbf{i}} = (12)(345)$. Then $\chi_{(3,2)}(C_{\mathbf{i}} = (x_1 - x_2)(x_1^2 + x_2^2)(x_1^3 + x_2^3)|_{(4,2)} = 1$.

Casimir value

Given a Young tableau R with n boxes, the quadratic Casimir of the representation is given by

$$(1) \qquad C_2(R) = nN + \frac{n(n-1)\chi_R(T_n)}{\chi_R(1)} - \frac{n^2}{N},$$

where $\chi_R(T_n)$ and $\chi_R(1) = d_R$ are characters of the symmetric group S_n in the representation associated with the Young tableau R. Here, $T_n =$ conjugacy class of elements in the symmetric group which have one cycle of length 2 and $n-2$ cycles of length 1. That is $T_{2,n} = \sum_{i<j}(ij) \in \mathbb{C}[S_n]$.

Casimir value

From long Moore:

$$
\begin{aligned}
C_2(R(Y)) &= Nn + 2\frac{\chi_{r(Y)}(T_{2,n})}{d_r(Y)}, \quad && U(N) \\
(2) \qquad &= Nn + 2\frac{\chi_{r(Y)}(T_{2,n})}{d_r(Y)} - \frac{n^2}{N}, \quad && SU(N)
\end{aligned}
$$

Or, in terms of the ℓ (shifted highest weight):

$$C_2(R(Y)) = \frac{N}{12}(N^2-1) + \sum_{j=1}^{N}(\ell_j - \frac{N-1}{2})^2.$$

Dimension relations

Let $r(Y)$ be S_n irrep. Then:

$$\dim R(Y) = (\frac{N^n \dim r(Y)}{n!})^m \frac{\chi_{r(Y)}(\Omega_n^m)}{\dim r(Y)}.$$

The dimension of the irrep of S_d corresponding to the partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ of d with k parts equals

$$\dim V_\lambda = \frac{d!}{\ell_1! \cdots \ell_k!} \Delta(\ell),$$

where as usual $\ell_j = \lambda_j + k - j$.

The dimension of the $GL(N, \mathbb{C})$ irrep $\mathbf{S}_\lambda V$ corresponding to $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N \geq 0)$ is (cf [FH], Thm 6.3)

$$\frac{\Delta(\ell)}{\Delta(1, 1, \cdots, 1)} = \Pi_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Appendix: Induced representations

Let $H \subset G$ be a closed subgroup and let $\sigma : H \to U(V)$ be a unitary representation. The induced representation $Ind_H^G V$ operators on the space

$W = \{V-$valued L^2 functions on $G : f(gh) = \sigma(h)^{-1} f$

The representation is

$$\Phi(g)f(x) = f(g^{-1}x).$$

A Yang-Mills question

The non-abelian Migdal formula shows that critical points of the YM functional are in 1-1 correspondence with irreps of the group $U(N)$ of the bundles. On the other hand, the topological types of bundles are parametrized by their top Chern class c_N (check). The top Chern class apparently corresponds to the number of boxes in Young diagrams associated to critical points of this topological type.