INVERSE SPECTRAL PROBLEM FOR ANALYTIC $(\mathbb{Z}/2\,\mathbb{Z})^n$-SYMMETRIC DOMAINS IN $\mathbb{R}^n$

HAMID HEZARI AND STEVE ZELDITCH

Abstract. We prove that bounded real analytic domains in $\mathbb{R}^n$ with the symmetries of an ellipsoid, and with one axis length fixed, are determined by their Dirichlet or Neumann eigenvalues among other bounded real analytic domains with the same symmetries and axis length. Some non-degeneracy conditions are also imposed on the class of domains. It follows that bounded, convex analytic domains are determined by their spectra among other such domains. This seems to be the first positive result on the well-known Kac problem, can one hear the shape of a drum?, in higher dimensions.

1. Introduction and the Statement of results

The purpose of this article is to prove that bounded analytic domains $\Omega \subset \mathbb{R}^n$ with $\pm$ mirror symmetries across all coordinate axes, and with one axis height fixed (and also satisfying some generic non-degeneracy conditions) are spectrally determined among other such domains. This inverse result (Theorem 1) gives a higher dimensional analogue of the main result of [Z2] that "bi-axisymmetric" real analytic plane domains are spectrally determined. To our knowledge, it is the first positive higher dimensional inverse spectral result for Euclidean domains which is not restricted to balls. Negative results (i.e. constructions of non-isometric isospectral pairs) are given in [U, GW, GWW] (see also [GS] for some non-Euclidean domains). Higher dimensional inverse results for semi-classical Schrödinger operators with similar symmetries have recently been proved in [GU, H].

As in [Z2, Z3, Z4, H], a key step in the proof of the Theorem 1 is to obtain a sufficiently explicit formula for the 'wave invariants' at iterates γ^r of a non-degenerate bouncing ball orbit. The formula, given in Theorem 2, is valid without any symmetry assumptions. A new feature of the proof, which makes it rather different from that in [Z3, Z4], is the construction and analysis of a microlocal monodromy operator associated to γ, inspired by the works of Sjöstrand-Zworski [SZ] and Cardoso-Popov [CP] (see also [ISZ]), but employing a layer potential analysis more closely related to that in [Z3, HZ, TZ].

1.1. Statement of results.

We consider the eigenvalue problem on the domain Ω with the Euclidean Laplacian Δ_Ω^B and with boundary conditions B:

$$(1)\qquad \begin{cases} \Delta_\Omega^B \varphi_j(x) = \lambda_j^2 \varphi_j(x), \quad \langle \varphi_i, \varphi_j \rangle = \delta_{ij}, \quad (x \in \Omega) \\ \\ B\varphi_j(y) = 0, \quad y \in \partial\Omega. \end{cases}$$

The boundary conditions could be either Dirichlet $B\varphi = \varphi|_{\partial\Omega}$, or Neumann $B\varphi = \partial_\nu \varphi|_{\partial\Omega}$ where ∂_ν is the interior unit normal.

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The $(\mathbb{Z}/2\,\mathbb{Z})^n$ symmetries of the title are the maps

$$\sigma_j : (x_1, \dots, x_n) \to (x_1, \dots, -x_j, x_{j+1}, \dots, x_n) \tag{2}$$

and we assume that they are isometries of Ω. The symmetry assumption implies that the intersections of the coordinate axes with Ω are projections of bouncing ball orbits preserved by the symmetries. We recall that a bouncing ball orbit γ is a 2-link periodic trajectory of the billiard flow, i.e. a reversible periodic billiard trajectory that bounces back and forth along a line segment orthogonal to the boundary at both endpoints. The endpoints of the projection to Ω of the bouncing ball orbit are fixed points of all but one of the isometries σ_j; the remaining one fixes the projected orbit setwise but interchanges the endpoints. We add the generic condition that at least one of these bouncing ball orbits is *non-degenerate* (see §1.1 for the conditions). We also fix the length $L_\gamma = 2L$ of this bouncing ball orbit γ.

We denote by $\mathcal{D}_L$ to be the class of all bounded real-analytic domains $\Omega \subset \mathbb{R}^n$ satisfying these assumptions. Thus, $\mathcal{D}_L$ is the class of domains such that:

(3)
$$\begin{cases} \text{(i)} \quad \sigma_j : \Omega \to \Omega \text{ is an isometry for all } j = 1, \dots, n; \\ \text{(ii) One of the coordinate axis bouncing ball orbits is non-degenerate and of length } 2L \\ \text{(iii) The lengths } 2rL \text{ of all iterates } \gamma^r (r = 1, 2, 3, \dots) \text{ have multiplicity one in } Lsp(\Omega); \\ \text{(iv) If } \gamma \text{ is elliptic and } \{e^{\pm i\alpha_1}, \dots e^{\pm i\alpha_{n-1}}\} \text{ are the eigenvalues of the Poincare map } \mathcal{P}_\gamma \text{, we} \\ \qquad \text{also require that } \{\alpha_1, \dots, \alpha_{n-1}\} \text{ are linearly independent over } \mathbb{Q}. \text{ We assume the same} \\ \qquad \text{independence condition in the Hyperbolic case or mixed cases.} \end{cases}$$

Let $\mathrm{Spec}_B(\Omega)$ denote the spectrum of the Laplacian Δ_Ω^B of the domain Ω with boundary conditions B (Dirichlet or Neumann).

THEOREM **1.** *For Dirichlet (or Neumann) boundary conditions B, the map $Spec_B : \mathcal{D}_L \mapsto \mathbb{R}_+^{\mathbf{N}}$ is 1-1.*

As a corollary, we obtain a result for convex analytic domains that does not require any length to be marked.

COROLLARY **1.** *Let $\mathcal{C}$ be the class of analytic convex domains with $(\mathbb{Z}/2\,\mathbb{Z})^n$ symmetry, such that the shortest closed billiard trajectory γ_0 is non-degenerate and satisfies the conditions (iii) and (iv) of (3). Then $Spec_B$: $\mathcal{C} \mapsto \mathbb{R}_+^{\mathbf{N}}$ is 1-1.*

This follows from Theorem 1 and a result of M. Ghomi [Gh] that the shortest closed trajectory of a centrally-symmetric convex domain is automatically a bouncing ball orbit. Hence the length of this orbit is self-marked, and it is not necessary to mark the length $L_\gamma = 2L$ of an invariant bouncing ball orbit γ.

1.2. **Formula for the wave invariants at a bouncing ball orbit.** As mentioned above, the main step in the proof is to obtain a concrete formula for the wave invariants $B_{\gamma^r,j}$ associated to the iterates γ^r of a symmetric bouncing ball orbit γ defined by one of the coordinate axes. (cf. 2.2). This result is of independent interest and is valid without any symmetry assumptions.

The wave invariants are defined in Definition 2.2. We do not actually study the trace of the wave group but an essentially equivalent semi-classical smoothed resolvent trace with spectral parameter along a logarithmic curve. The trace asymptotics are eventually reduced to those of a boundary integral operator in Proposition 5.6 and Corollary 5.7, and then to the stationary phase asymptotics of a certain oscillatory integral in Theorem 6.3. The method and results give a higher dimensional generalization of the analogous results of [Z3, Z4]. Some overlap in the arguments from the two-dimensional case is inevitable; we have tried to minimize the overlap, but it is necessary to give complete details on the formulae in n dimensions since they differ in numerous ways from the two-dimensional case.

To state the formula for $B_{\gamma^r,j}$, we introduce some notations and conventions. We align axes so that the bouncing ball orbit γ is a vertical segment of length L with endpoints at $A = (0, \frac{L}{2})$ and $B = (0, -\frac{L}{2})$, where 0 denotes the origin in the orthogonal $x' = (x^1, \dots, x^{n-1}) \in \mathbb{R}^{n-1}$ plane. In a metric tube $T_\epsilon(\overline{AB})$ of radius ϵ around γ, we may locally express $\partial\Omega = \partial\Omega^+ \cup \partial\Omega^-$ as the union of two graphs over a ball $B_\epsilon(0)$ around 0 in the x'-hyperplane, namely

$$\partial\Omega^+ = \{x^n = f_+(x'), \quad |x'| \le \epsilon\}, \ \ \partial\Omega^- = \{x^n = f_-(x'), \quad |x'| \le \epsilon\}. \tag{4}$$

We will use the standard shorthand notations for multi-indices, i.e. $\vec{\gamma} = (\gamma_1, \dots \gamma_{n-1})$, $|\vec{\gamma}| = \gamma_1 + \dots + \gamma_n$, $\vec{X}^{\vec{\gamma}} = X_1^{\gamma_1} \dots X_{n-1}^{\gamma_{n-1}}$, and by $\overrightarrow{h^{pq}_{+,2r}}$, we mean the $(n-1)$-vector

$$\overrightarrow{h^{pq}_{+,2r}} = (h^{11,pq}_{+,2r}, h^{22,pq}_{+,2r}, \dots, h^{(n-1,n-1),pq}_{+,2r}),$$

where $[h^{ij,pq}_{+,2r}]_{1 \le i,j \le n-1, 1 \le p,q \le 2r}$ is the inverse Hessian matrix of the length functional $\mathcal{L}_\pm(x'_1, \dots, x'_{2r})$ given in (10).

The following generalizes Theorem 5.1 in [Z3] from two to higher dimensions.

THEOREM **2.** *Let Ω be a smooth domain with a bouncing ball orbit γ of length $2L$ and let $B_{\gamma^r,j}$ be the wave invariants associated to γ^r (see cf. 2.2). Then for each $r = 1, 2, \dots,$ and j there exists a polynomial $P_{r,j}$ such that:*

1. *$B_{\gamma^r,j} = P_{r,j}(\{D^{|\gamma|}_{\vec{\gamma}} f_+(0)\}, \{D^{|\gamma|}_{\vec{\gamma}} f_-(0)\})$ with $|\gamma| \le 2j+2$; i.e. the highest order of derivatives appearing in $B_{\gamma^r,j}$ is $2j+2$*
2. *In the polynomial expansion of $B_{\gamma^r,j}$ the Taylor coefficients of order $2j+2$ appear in the form $\{D^{2j+2}_{2\vec{\gamma}} f_\pm(0)\}$.*
3. *$B_{\gamma^r,0}$ is only a function of r, L and n, and for $j \ge 1$*

$$B_{\gamma^r,j} = \frac{B_{\gamma^r,0}}{(2i)^{j+1}} \sum_{|\gamma|=j+1} \frac{r}{\vec{\gamma}!} \{ (\overrightarrow{h^{11}_{+,2r}})^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f_+(0) - (\overrightarrow{h^{11}_{-,2r}})^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f_-(0) \}$$

$$+ R_{2r,j}(\mathcal{J}^{2j+1} f_+(0), \mathcal{J}^{2j+1} f_-(0)),$$

where the remainder $R_{2r,j}(\mathcal{J}^{2j+1} f_+(0), \mathcal{J}^{2j+1} f_-(0))$ is a polynomial in the designated jet of $f_\pm$.

4. *In the $(\mathbb{Z}/2\,\mathbb{Z})$- symmetric case, where $f_+ = f = -f_-$, we have the simplified formula*

$$B_{\gamma^r,j} = \frac{B_{\gamma^r,0}}{(2i)^{j+1}} \sum_{|\vec{\gamma}|=j+1} \frac{r}{\vec{\gamma}!} \left(\frac{1}{\sin \frac{\vec{\alpha}}{2}} \cot \frac{r\vec{\alpha}}{2} \right)^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f(0) + R_{2r,j}(\mathcal{J}^{2j+1} f(0)).$$

5. *In the $(\mathbb{Z}/2\,\mathbb{Z})^n$- symmetric case, formula (4) holds with remainder in $R_{2r,j}(\mathcal{J}^{2j} f(0))$.*

In the above notation, the $(\mathbb{Z}/2\,\mathbb{Z})^n$-symmetry assumptions in (5) are that

$$f_+(x') = -f_-(x'), \ \ f_\pm(\sigma_j(x')) = f_\pm(x'), \tag{5}$$

where σ_j denotes the reflections in the coordinate hyperplanes of $\mathbb{R}^{n-1}$. The first assumption implies that there exists a function $f(x')$ so that the top of the domain is defined by $x_n = f(x')$ and the bottom is defined by $x_n = -f(x')$. The further symmetry assumptions then say that f is an even function in every variable x^i, $1 \leq i \leq n-1$, i.e.

$$f(x^1, \dots, x^{n-1}) = F((x^1)^2, \dots, (x^{n-1})^2), \qquad F \in C^\omega(\mathbb{R}^{n-1}). \tag{6}$$

1.3. **Determining Taylor coefficients from wave invariants.** The final step is to determine the defining function of the domain from the wave invariants at the bouncing ball orbit. When γ is invariant under the $(\mathbb{Z}/2\,\mathbb{Z})^n$ symmetries, it is quite simple to prove that the Taylor coefficients of the defining function of the domain at the endpoints of γ can be recovered from the wave invariants (see §7.6). In fact, our method could be extended to show that analytic domains with fewer symmetries are spectrally determined as in [Z3, H], but for the sake of brevity we do not prove that here. The proof builds on the methods of [Z3, H].

1.4. **Discussion and Comparison of Methods.** We obtain the formulae for the wave invariants by applying the stationary phase method to the trace of a well-constructed parametrix for this monodromy operator. A secondary purpose of this article is to connect the very conceptual but somewhat abstract monodromy method of [ISZ, SZ] with the methods of [Z1, Z2, Z3]. The articles [Z1, Z2] implicitly used the monodromy approach in the form given in [BBa, L, LT]. In this article, we construct the monodromy operator explicitly in terms of layer potentials, using in part the methods of [CP] and in part those of [Z3, HZ]. The monodromy approach connects nicely with the 'Balian-Bloch' approach of [Z3] and simplifies remainder estimates for the Balian-Bloch (i.e. Neumann) expansion of the resolvent.

In calculating the trace asymptotics, we do not put the monodromy operator into normal form, but rather apply a direct stationary phase analysis to the parametrix. Terms of the stationary phase expansion correspond to Feynman diagrams and the main idea (as in [Z3]) is to isolate the diagrams which are necessary and sufficient to determine the Taylor coefficients of the boundary defining function at the endpoints of γ from the wave trace invariants of iterates of γ. In this $(\mathbb{Z}/2\,\mathbb{Z})^n$ symmetric case, there is a unique such diagram and that is why the symmetries simplify the problem. It is an interesting but difficult problem to 'invert' the spectrum when some or all of the symmetries are absent.

An alternative to the approach of this article is to use quantum Birkhoff normal forms around the bouncing ball orbit as in [G, G2, Z1, Z2, ISZ, SZ]. It suffices to prove the abstract result that the quantum normal form of the Laplacian or wave group at the invariant bouncing ball orbit, hence that the classical Birkhoff normal form of the Poincaré map, is a spectral invariant. At that point, one could generalize the result of Y.Colin de Verdière [CV] that the classical normal form determines the Taylor coefficients of f at the endpoints of the bouncing ball orbit when f has the $(\mathbb{Z}/2\mathbb{Z})^n$ symmetries. We plan to carry out the details in a follow-up to this article. The methods of this article go further, since Theorem 2 determines much more than the classical Birkhoff normal form.

It would be interesting to obtain a more direct connection between the the 'normal forms' approach and the 'parametrix approach'. In general terms, normal forms for Hamiltonians and for canonical transformations belong to the canonical Hamiltonian formulation of quantum mechanics, while parametrix constructions, stationary phase methods and Feynman diagrams belong to the Lagrangian or path integral approach. Normal forms are of course canonical, while parametrices are not: there are many possible parametrices (finite dimensional approximations to path integrals), and in the inverse problem it is essential to construct computable ones. The two approaches are dual, and although they contain the same information, it is formatted in different ways. In particular, the two approaches highlight different features of the geometry and dynamics.

At the present time, explicit calculations and spectral inversion for boundary problems have only been carried out in the Lagrangian approach, despite the existence of a quantum normal form along bouncing ball orbits [Z2]. In the simpler setting of semi-classical Schrödinger operators at equilibrium points, one may compare the normal forms approach of [GU, CVG] to the Lagrangian approach of [H]. In this inverse problem, one has a one-parameter family of isospectral operators depending on a Planck's constant h, whereas in the boundary problem one has only one operator and spectrum to work with. The Lagrangian calculations in [H] reproduced the inverse results of [GU, CVG], and gave stronger ones where some of the symmetries were removed. It directly gives formulae for wave invariants, which are linear combinations of normal form invariants.

It is interesting to observe that formula in Theorem 2 is very similar to the formula in [H] (Theorem 2.1) for the wave invariants at an equilibrium point for a Schrödinger operator on $\mathbb{R}^n$ with a unique equilibrium point at $(x, \xi) = (0, 0)$. This perhaps indicates a similarity between the quantum normal form of the Schrödinger operator at the equilibrium point and that of the Laplacian at a bouncing ball orbit. The formula is also similar to a trace asymptotics formula of T. Christiansen for an inverse problem for wave-guides [Chr], but that is less surprising.

Finally, we mention that the methods of this paper have further applications. In a work in progress [HeZ], we use the wave invariants to prove a certain spectral rigidity result for analytic deformations of an ellipse.

2. Background

In this section, we go over the basic set-up of the problem. It is very similar to that of [Z3] but requires some higher dimensional generalizations. We use the same notation as in [Z3] and refer there for many details.

2.1. **Billiard map.** The billiard map β is defined on

$$B^*\partial\Omega = \{(y, \eta);\ y \in \partial\Omega,\ \eta \in T^*\partial\Omega,\ |\eta| \leq 1\}$$

as follows: given $(y, \eta) \in B^*\partial\Omega$, with $|\eta| \leq 1$, let $(y, \zeta) \in S^*\Omega$ be the unique inward-pointing unit covector at y which projects to (y, η) under the map $T^*_{\partial\Omega}\overline{\Omega} \to T^*\partial\Omega$. Then follow the geodesic (straight line) determined by (y, ζ) to the first place it intersects the boundary again; let $y' \in \partial\Omega$ denote this first intersection. (If $|\eta| = 1$, then we let $y' = y$.) Denoting the inward unit normal vector at y' by $\nu_{y'}$, we let $\zeta' = \zeta + 2(\zeta \cdot \nu_{y'})\nu_{y'}$ be the direction of the geodesic after elastic reflection at y', and let η' be the projection of ζ' to $B^*_{y'}\partial\Omega$. Then we define

$$\beta(y, \eta) = (y', \eta').$$

The billiard map is a symplectic, hence measure preserving, map with respect to the standard symplectic form on $T^*\partial\Omega$. We denote its graph of β by

$$C_{\text{billiard}} := \text{graph}\,\beta \equiv \{(\beta(z), z) \mid z \in B^*\partial\Omega\}. \tag{7}$$

In the case of a convex domain,

$$C_{\text{billiard}} = \Gamma_d := \{\big(y, -\nabla_y d(y,y'), y', \nabla_{y'} d(y,y')\big)\}, \tag{8}$$

i.e. the Euclidean distance function $d(y,y')$ is a generating function for β. For non-convex domains, this graph is larger due to 'ghost' billiard trajectories which exit and re-enter Ω but satisfy the reflection law of equal angles at each intersection point. Such ghost orbits are the price one pays for using a parametrix and distance function $d(y,y')$ defined on the ambient space $\mathbb{R}^n$.

2.2. **Length functional.** We define the length functional on $(\partial\Omega)^m$ (the Cartesian product), by

$$L(y_1,\ldots,y_m) = |y_1 - y_2| + \cdots + |y_{m-1} - y_m| + |y_m - y_1|. \tag{9}$$

Then L is a smooth away from the 'large diagonals' $\Delta_{p,p+1} := \{y_p = y_{p+1}\}$. The condition that $dL = 0$ is the classical condition that each 2-link defined by the triplet (y_{p-1}, y_p, y_{p+1}) makes equal angles with the normal at y_p. Hence a smooth critical point corresponds to a closed m-link billiard trajectory. See for instance §2.1 of [PS].

2.2.1. *Length functional in graph coordinates near the iterates of a bouncing ball orbit.* A bouncing ball orbit γ is a 2-link periodic trajectory of the billiard flow, i.e. a reversible periodic billiard trajectory that bounces back and forth along a line segment orthogonal to the boundary at both endpoints. As in the Introduction we orient Ω so that $\overline{AB}$ lies along the vertical x_n axis, with $A = (0, \frac{L}{2}), B = (0, -\frac{L}{2})$. We parameterize the boundary locally as two graphs $x_n = f_\pm(x')$ over the x'-hyperplane. Thus, in a small tube $T_\epsilon(\gamma)$ around $\overline{AB}$, the boundary consists of two components, which are graphs of the form $y = f_+(x')$ near A and $y = f_-(x')$ near B.

We then define the length functionals in Cartesian coordinates for the two possible orientations of the rth iterate of a bouncing ball orbit by

$$\mathcal{L}_\pm(x'_1,\ldots,x'_{2r}) = \sum_{p=1}^{2r} \sqrt{(x'_{p+1} - x'_p)^2 + (f_{w_\pm(p+1)}(x'_{p+1}) - f_{w_\pm(p)}(x'_p))^2}. \tag{10}$$

Here, $w_\pm : \mathbb{Z}_{2r} \to \{\pm\}$, where $w_+(p)$ (resp. $w_-(p)$) alternates sign starting with $w_+(1) = +$ (resp. $w_-(1) = -$). Obviously the point $(x'_1,\ldots,x'_{2r}) = (0,\ldots,0)$, corresponding to the r-the iteration of the bouncing ball orbit, is a critical point of $\mathcal{L}_\pm$.

2.3. **Resolvent and Wave group.** By the Dirichlet Laplacian Δ_Ω we mean the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ with domain $\{u \in H_0^1(\Omega) : \Delta u \in L^2\}$; thus, in our notation, Δ_Ω is a negative operator. The resolvent $R_\Omega^D(k)$ of the Laplacian Δ_Ω on Ω with Dirichlet boundary conditions is the family of operators on $L^2(\Omega)$ defined for $k \in \mathbb{C}, \Im k > 0$ by

$$R_\Omega^D(k) = -(\Delta_\Omega + k^2)^{-1}, \qquad \Im k > 0.$$

The resolvent kernel, which we refer to as the *Dirichlet Green's function* $G^D_\Omega(k,x,y)$ of $\Omega \subset \mathbb{R}^2$, is by definition the solution of the boundary problem:

$$\begin{cases} (\Delta_x + k^2) G^D_\Omega(k,x,y) = -\delta(x-y), \quad (x,y \in \Omega) \\ \\ G^D_\Omega(k,x,y) = 0, \quad x \in \partial\Omega. \end{cases} \tag{11}$$

The resolvent may be expressed in terms of the even wave operator $E_\Omega(t) = \cos(t\sqrt{-\Delta_\Omega})$ as

$$R^D_\Omega(k) = \frac{1}{k}\int_0^\infty e^{ikt} E_\Omega(t)\,dt, \qquad (\Im k > 0) \tag{12}$$

In this paper we will consider the resolvent $R^D_\Omega(k)$ along the logarithmic ray $k = \lambda + i\tau\log\lambda$, where $\lambda > 1$ and $\tau \in \mathbb{R}^+$.

Given $\hat{\rho} \in C_0^\infty(\mathbb{R}^+)$, we define the smoothed resolvent $R^D_{\Omega,\rho}(k)$ by

$$R^D_{\Omega,\rho}(k) := \rho * (\mu R^D_\Omega(\mu)) = \int_{\mathbb{R}} \rho(k-\mu)(\mu R^D_\Omega(\mu))\, d\mu. \tag{13}$$

We note that $\rho(k-\mu)$ is well-defined since ρ is an entire function. Let us discuss in what sense the integral in (13) is defined. We notice that since $\mu \in \mathbb{R}$, we have defined the resolvent $R^D_\Omega(\mu)$ by $R^D_\Omega(\mu + i0^+)$. Hence we can write

$$\begin{aligned} R^D_{\Omega,\rho}(k) &= \lim_{\epsilon\to 0^+} \textstyle\int_{\mathbb{R}} \rho(k-\mu)\mu R^D_\Omega(\mu + i\epsilon)\, d\mu \\ &= -\lim_{\epsilon\to 0^+} \textstyle\int_{k-\mathbb{R}} \rho(\mu')(k-\mu') R^D_\Omega(k + i\epsilon - \mu')\ d\mu' \\ &= \textstyle\int_{\mathbb{R}} \rho(\mu)(k-\mu) R^D_\Omega(k-\mu)\ d\mu, \end{aligned}$$

where the last equality is obtained by taking an appropriate contour. Thus we can also take this last integral as our definition of the smoothed resolvent.

Now by (12) we can rewrite $R^D_{\Omega,\rho}(k)$ in terms of the wave kernel as:

$$\begin{aligned} R^D_{\Omega,\rho}(k) &= \textstyle\int_0^\infty \int_{\mathbb{R}} \rho(\mu) e^{i(k-\mu)t} E_\Omega(t)\,dt\,d\mu \\ &= \textstyle\int_0^\infty \hat{\rho}(t) e^{ikt} E_\Omega(t)\,dt. \end{aligned} \tag{14}$$

The Poisson formula for manifolds with boundary [AM, GM, PS] gives the existence of a singularity expansion for the trace of $E_\Omega(t)$ near a transversal reflecting ray. If we substitute this singularity expansion into the trace of (14) we obtain an asymptotic expansion in inverse powers of k for the smoothed resolvent trace:

Theorem 2.1. [AM, GM, PS] *Assume that γ is a non-degenerate periodic reflecting ray, and let $\hat{\rho} \in C_0^\infty(L_\gamma - \epsilon, L_\gamma + \epsilon)$, such that $\hat{\rho} \equiv 1$ on $(L_\gamma - \epsilon/2, L_\gamma + \epsilon/2)$ and with no other lengths in its support. Then for $k = \lambda + i\tau\log\lambda$ with $\tau \in \mathbb{R}^+$, $Tr R^D_{\Omega,\rho}(k)$ admits a complete asymptotic expansion of the form*

$$Tr R^D_{\Omega,\rho}(k) \sim \mathcal{D}_{D,\gamma}(k) \sum_{j=0}^\infty B_{\gamma,j} k^{-j}, \quad \Re k \to \infty, \tag{15}$$

where

- $\mathcal{D}_{D,\gamma}(k)$ *is the symplectic pre-factor*
$$\mathcal{D}_{D,\gamma}(k) = C_0\, \epsilon_D(\gamma) \frac{e^{ikL_\gamma} e^{i\frac{\pi}{4} m_\gamma}}{\sqrt{|\det(I - P_\gamma)|}};$$
- P_γ *is the Poincaré map associated to* γ*;*
- $\epsilon_D(\gamma)$ *is the signed number of intersections of* γ *with* $\partial\Omega$*;*
- m_γ *is the Maslov index of* γ*;*
- C_0 *is a universal constant (e.g. factors of* 2π*).*

Definition **2.2.** *The coefficients* $B_{\gamma,j}$ *are called the wave trace invariants (Balian-Bloch wave invariants) associated to the periodic orbit* γ*.*

3. Resolvent and the layer potentials

The method of layer potentials ([T] II, §7. 11) solves (11) in terms of the 'layer potentials' $G_0(k,x,y), \partial_{\nu_y} G_0(k,x,y) \in \mathcal{D}'(\Omega \times \partial\Omega)$, where ν is the interior unit normal to Ω, and $\partial_\nu = \nu \cdot \nabla$, and where $G_0(k,x,y)$ is the 'free' Green's function of $\mathbb{R}^n$, i.e. of the kernel of the free resolvent $-(\Delta_0 + k^2)^{-1}$ of the Laplacian Δ_0 on $\mathbb{R}^n$. A key point, first recognized in [BB1, BB2] and put on a rigorous mathematical basis in [Z3, HZ], is that the layer potentials are semi-classical (i.e. non-homogeneous) Lagrangian distributions in the k parameter, with additional homogeneous singularities on the diagonal. In effect we wish to make use of the first property and suppress the second. This will be explained in §3.1.

First let us briefly recall the method of layer potentials. The free outgoing Green's function in dimension n is given by:

$$G_0(k,x,y) = \frac{i}{4} k^{n-2} (2\pi k|x-y|)^{-(n-2)/2}\, \mathrm{Ha}^{(1)}_{n/2-1}(k|x-y|). \tag{16}$$

In general, the Hankel function of index ν has the integral representation ([T], Chapter 3, §6)

$$\mathrm{Ha}^{(1)}_\nu(z) \;=\; (\tfrac{2}{\pi z})^{1/2} \tfrac{e^{i(z-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-s} s^{-1/2} (1 - \tfrac{s}{2iz})^{\nu-1/2} ds. \tag{17}$$

The single and double *layer potentials*, as operators from the boundary $\partial\Omega$ to the interior Ω, are then respectively defined by

$$\begin{cases} \mathcal{S}\ell(k) f(x) = \int_{\partial\Omega} G_0(k,x,y) f(y) dS(y), & x \in \Omega, \\ \mathcal{D}\ell(k) f(x) = \int_{\partial\Omega} 2\, \partial_{\nu_y} G_0(k,x,y) f(y) dS(y), & x \in \Omega. \end{cases} \tag{18}$$

Similarly, for a function f on $\partial\Omega$, the *boundary layer potentials* $S(k)$ and $N(k)$, as operators from the boundary $\partial\Omega$ to itself are denoted by:

$$\begin{cases} S(k) f(x) = \int_{\partial\Omega} G_0(k,x,y) f(y) dS(y), & x \in \partial\Omega, \\ N(k) f(x) = \int_{\partial\Omega} 2\, \partial_{\nu_y} G_0(k,x,y) f(y) dS(y), & x \in \partial\Omega. \end{cases} \tag{19}$$

Given any function g on $\mathbb{R}^n \backslash \partial\Omega$ and any $x \in \partial\Omega$, we denote by $g_+(x)$ (resp. $g_-(x)$) the limits of $g(w)$ as $w \to x \in \partial\Omega$ from $w \in \Omega$ (resp. $w \in \mathbb{R}^n \backslash \bar{\Omega}$). The layer potentials and the boundary layer potentials introduced above are related by the following

$$(\mathcal{S}\ell(k)f)_+(x) = (\mathcal{S}\ell(k)f)_-(x) = S(k)f(x),$$
$$(\mathcal{D}\ell(k)f)_\pm(x) = (\pm I + N(k))f(x), \qquad \text{(The jump formula)}. \tag{20}$$

Using the above notation we also have the following interesting formula of Fredholm-Neumann

$$R^D_\Omega(k) - R_0(k) = -\mathcal{D}\ell(k)(I+N(k))^{-1}\mathcal{S}\ell(k)^t, \tag{21}$$

where $R_0(k)$ is the free resolvent. This is true because the kernels of both sides of the equation are solutions to the Helmholtz equation and also the restrictions to the boundary of these kernels are the same by the jump formula (20). The formula follows by the uniqueness of the solutions of the Helmholtz equation.

3.1. **Structure of the operator** $N(k)$**.** We now state the precise sense in which $N(k)$ is a semi-classical Fourier integral operator quantizing the billiard map of $\partial\Omega$ when $k = \lambda + i\tau\log\lambda$ and $\lambda = \Re k \to \infty$. It additionally has homogeneous singularities on the diagonal. The discussion here is adapted from [HZ]. Note that here our semiclassical parameter is $\frac{1}{k}$ which is a complex parameter.

We denote $S^*\partial\Omega = \{(y,\eta) \in T^*\partial\Omega;\ |\eta| = 1\}$ and define the diagonal set

$$\Delta_{S^*\partial\Omega} = \{(z,z);\ z \in S^*\partial\Omega\} \subset T^*\partial\Omega \times T^*\partial\Omega. \tag{22}$$

Proposition **3.1.** [HZ] *Assume that* Ω *is a smooth domain. Let* U *be any neighborhood of* $\Delta_{S^*\partial\Omega}$*. Then there is a decomposition* $N(k)$ *as*

$$N(k) = N_0(k) + N_1(k) + N_2(k),$$

where $N_0(k)$ *is a pseudodifferential operator of order* -1 *,* $N_1(k)$ *is a semi-classical Fourier integral operator of order zero associated with the canonical relation* Γ_d *(cf. (8)), and* $N_2(k)$ *has operator wavefront set contained in* U*.*

This proposition is valid whether or not Ω is convex, but in the convex case $\Gamma_d = C_{\text{billiard}}$ is the graph of the billiard map (cf §2.1).

From the integral formula (17), the Hankel function $\mathrm{Ha}^{(1)}_{n/2-1}(z)$ is conormal at $z=0$ and as $z \to \infty$,

$$a(z) := e^{-iz}\,\mathrm{Ha}^{(1)}_{n/2-1}(z) \sim z^{-1/2}\sum_{j=0}^{\infty} a_j z^{-j}. \tag{23}$$

The kernel of the single-layer potential then has the form

$$S(k,x,y) = G_0(k,x,y) = Ck^{n-2}(k|x-y|)^{-(n-2)/2}\,\mathrm{Ha}^{(1)}_{n/2-1}(k|x-y|).$$

For the double layer potential it follows from the identity

$$\frac{d}{dz}\,\mathrm{Ha}^{(1)}_\nu(z) = \frac{\nu}{z}\,\mathrm{Ha}^{(1)}_\nu(z) - \mathrm{Ha}^{(1)}_{\nu+1}(z),$$

that

$$N(k,x,y) = 2\,\partial_{\nu_y}G_0(k,x,y) = Ck^{n-1}(k|x-y|)^{-(n-2)/2}\,\mathrm{Ha}^{(1)}_{n/2}(k|x-y|)\,\langle\frac{x-y}{|x-y|},\nu_y\rangle.$$

We now introduce cutoff functions as in [HZ]. When $k = \lambda + i\tau \log \lambda$, we put

$$1 = \varphi_1(|x-y|) + \varphi_0(|k|^{3/4}|x-y|) + \varphi_2(|x-y|, |k|)$$

where $\varphi_1(t)$ is supported in $t \geq t_0$ for some sufficiently small $t_0 > 0$ (see [HZ] for the choice of this t_0), and $\varphi_0(t)$ is equal to 1 for $t \leq 1$ and equal to 0 for $t \geq 2$. (The power 3/4 in φ_0 could be replaced by any other power strictly between 1/2 and 1). We define the operator $N_1(k)$ to be the one with the kernel $\varphi_1(|x-y|)N(k,x,y)$ which has the form

$$N_1(k,x,y) = Ck^{(n-1)} e^{ik|x-y|} \varphi_1(|x-y|) a_1(k|x-y|)\, \langle \frac{x-y}{|x-y|}, \nu_y \rangle, \tag{24}$$

where $a_1(z) = z^{-\frac{n-2}{2}} a(z)$ has an expansion in inverse powers of z as $z \to \infty$ (see (23)), with leading term $z^{-(n-1)/2}$. The operator $N_1(k)$ is manifestly a semiclassical FIO of order 0 with the phase function $d(x,y) = |x-y|$, and thus associated with the canonical relation Γ_d (cf. (8)). The operators $N_0(k)$ and $N_2(k)$ are constructed similarly from the cut offs φ_0 and φ_2 respectively. See [HZ] for the details.

4. Monodromy operators and boundary integral operators

It simplifies the resolvent trace calculation considerably to reduce it to a boundary trace. In [Z4, Z3] this was done by taking the direct sum $R^D_{\Omega,\rho}(k) \oplus R^N_{\Omega^c,\rho}(k)$ of the interior Dirichlet and exterior Neumann resolvents, and verifying that

$$Tr\big(R^D_{\Omega,\rho}(k) \oplus R^N_{\Omega^c,\rho}(k) - R_{0,\rho}(k)\big) = Tr_{\partial\Omega}\big(\rho * \frac{d}{dk} \log\det(I + N(k))\big).$$

The construction for the interior Neumann resolvent and exterior Dirichlet resolvent is essentially the same and is omitted for brevity. This relation was stated in the physics literature, and we refer to [Z4] for references. In this article, we take a related but somewhat different method to reduce the trace to the boundary using monodromy operator ideas similar to those of Cardoso-Popov [CP] and of Sjöstrand-Zworski [SZ]. A novel feature is that we relate the monodromy operators to the boundary integral operators $N_1(k)$ and $N_0(k)$.

Before going into the details, let us note some motivating ideas. First, both the monodromy operator $M(k)$ and the boundary integral operator $N_1(k)$ are quantizations of the billiard map in the sense of being semi-classical Fourier integral operators whose canonical relation is C_{billiard}. This suggests that they must be closely related. However, $M(k)$ is microlocally constructed while $N_1(k)$ is global. Further, $N_1(k)$ is just a piece of $N(k)$, which has the more complicated structure described in §3.1, and in the boundary reduction $N(k)$ is the primary object. On the other hand, there is a simple exact formula for $N(k)$ while $M(k)$ is only known through microlocal conjugation to normal form. Hence our purpose here is to construct a monodromy operator resembling $M(k)$ using $N(k)$ and the layer potentials. In doing so, we follow the approach to monodromy operators of [CP]. Since the resulting monodromy operator does not seem to arise from the abstract set-up of Grushin reductions, the proof that the trace reduces to the boundary is not the same as in [SZ].

4.1. **Definition of the monodromy operator.**

In this section we will use the basic terminologies in semi-classical analysis such as semi-classical pseudodifferential operators, Fourier integral operators and semi-classical wave front sets. We refer to [GS, EZ, SZ, Al] for all the definitions and properties. We just mention that in the following sections when we write

$T(k) \sim S(k)$, for two operators $T(k)$ and $S(k)$, we mean $T(k)-S(k)$ is a negligible (residual) operator in the sense that its kernel is of order $O(k^{-\infty})$ in all C^s norms.

Let γ be an m-link periodic reflecting ray, with vertices at v_j in $\partial\Omega$, where there are $m+1$ vertices and $v_{m+1} = v_1$. Let η_j be the projection to $B^*_{v_j}\partial\Omega$ of the direction of the ray γ where it hits the boundary at v_j. We denote

$$\partial\gamma := \{(v_1, \eta_1), \dots (v_m, \eta_m)\}$$

Let Γ_j be microlocal neighborhoods of (v_j, η_j) in $B^*\partial\Omega = \{(y, \eta) | y \in \partial\Omega,\ \eta \in T^*\partial\Omega, |\eta| \leq 1\}$. We then define a microlocalization to Γ_j of the double layer potential operator by

$$H_j(k) := \mathcal{D}\ell(k)\Psi_{\Gamma_j}(k) :\ C(\partial\Omega) \to C(\Omega),$$

where $\Psi_{\Gamma_j}(k)$ is a k-pseudodifferential operator microsupported in Γ_j. In fact we will choose,

$$\Psi_{\Gamma_j}(k) = (I + N_0(k))^{-1}\chi_{\Gamma_j}(k),$$

where χ_{Γ_j} is a microlocal cut off supported in Γ_j. More precisely $\chi_{\Gamma_j}(k) = Op_{\frac{1}{k}}(a_j(y, \eta))$, where $\mathrm{supp}(a_j(y, \eta)) \subset \Gamma_j$ and $a_j(y, \eta) = 1$ in a neighborhood of (v_j, η_j). Here $(I + N_0(k))^{-1}$ is a parametrix for $I+N_0(k)$. Notice a parametrix exists because $N_0(k)$ is a pseudodifferential of order -1.

Clearly, by the jump formula (20), for any $u \in C(\partial\Omega)$,

$$\begin{cases} (-\Delta - k^2)H_j(k)u = 0, \\ \\ H_j(k)(u)|_{\partial\Omega} = (I + N(k))\Psi_{\Gamma_j}(k)u. \end{cases}$$

Next we put for every $1 \leq i \leq m$,

(25)
$$P_i(k) = H_i(k) - H_{i+1}(k)r_{\partial\Omega}H_i(k) + \cdots + (-1)^{m-1}H_{i+(m-1)}(k)r_{\partial\Omega}H_{i+(m-2)}(k)\cdots r_{\partial\Omega}H_i(k),$$

where $r_{\partial\Omega} : C(\Omega) \to C(\partial\Omega)$ is the operator of restriction to the boundary. Also notice by our notation the indices $j + m$ and j are identified. Then we define

$$P(k) = \frac{1}{m}\sum_{i=1}^{m} P_i(k). \tag{26}$$

Proposition 4.1. *We have:* $(-\Delta - k^2)P(k)u = 0$.

Proof. In fact for every i we have $(-\Delta-k^2)P_i(k)u = 0$. We observe that $P_i(k) = \mathcal{D}\ell(k)Q_i(k)$ with

(27)
$$Q_i(k) = \Psi_{\Gamma_i} - \Psi_{\Gamma_{i+2}}\big[(I + N(k))\Psi_{\Gamma_{i+1}}\big] + \cdots + (-1)^{m-1}\Psi_{\Gamma_{i+m-1}}\big[(I + N(k))\Psi_{\Gamma_{i+m-2}}\cdots(I + N(k))\Psi_{\Gamma_i}\big].$$

The statement follows since $\mathcal{D}\ell(k)$ maps $C(\partial\Omega)$ into the solutions of the Helmholtz equation. □

We now make a couple of useful technical observations:

Proposition 4.2. *We have:*

$$r_{\partial\Omega}\, H_j(k) \sim (I + N_0(k) + N_1(k))\Psi_{\Gamma_j}.$$

Proof. By the jump formula we know that $r_{\partial\Omega} H_j(k) = (I + N(k))\Psi_{\Gamma_j}$. The missing term N_2 has its wave front set contained in $U \times U$ where U is a small neighborhood $S^*\partial\Omega$. But then $WF'(N_2) \circ WF'(\Psi_{\Gamma_j}) = \emptyset$. So this term of the composition is negligible in k. □

PROPOSITION **4.3.** *For all $1 \leq j < m$, we have:*

$$\Psi_{\Gamma_{j+1}}\big[(I + N(k))\Psi_{\Gamma_j} \cdots (I + N(k))\Psi_{\Gamma_1}\big] \sim \Psi_{\Gamma_{j+1}}\big[N_1(k)\Psi_{\Gamma_j} \cdots N_1(k)\Psi_{\Gamma_1}\big].$$

Proof. We argue inductively. For $j = 1$, using the proof of Proposition 4.2 we have

$$\Psi_{\Gamma_2}\big[(I + N(k))\Psi_{\Gamma_1}\big] \sim \Psi_{\Gamma_2} N_1(k)\Psi_{\Gamma_1} + \Psi_{\Gamma_2}\big[(I + N_0(k))\Psi_{\Gamma_1}\big].$$

But the last term in the expression above is negligible in k, as the two semi-classical pseudodifferentials Ψ_{Γ_2} and $\big[(I + N_0(k))\Psi_{\Gamma_1}\big]$ are microsupported in the two disjoint open sets Γ_2 and Γ_1 respectively. Now assume the statement is true for $j - 1$. Then we write

$$\Psi_{\Gamma_{j+1}}\big[(I + N(k))\Psi_{\Gamma_j} \cdots (I + N(k))\Psi_{\Gamma_1}\big] \sim \Psi_{\Gamma_{j+1}}\big[N_1(k)\Psi_{\Gamma_j} \cdots N_1(k)\Psi_{\Gamma_1}\big]$$

$$+\Psi_{\Gamma_{j+1}}(I+N_0(k))\Psi_{\Gamma_j}\big[N_1\Psi_{\Gamma_{j-1}} \cdots N_1(k)\Psi_{\Gamma_1}\big].$$

Similarly, $\Psi_{\Gamma_{j+1}}(I + N_0(k))\Psi_{\Gamma_j}$ is negligible in k and the second term above is negligible. □

Next we have the following important proposition which implicitly defines the monodromy operator:

PROPOSITION **4.4.** *We have*

$$r_{\partial\Omega}P(k) \sim I + M(k), \qquad (\textit{microlocally near } \partial\gamma)$$

where

$$M(k) = \frac{1}{m}\sum_{i=1}^{m}(-1)^{m-1}N_1(k)\Psi_{\Gamma_{i+m-1}} \cdots N_1(k)\Psi_{\Gamma_i}. \tag{28}$$

DEFINITION **4.5.** *The monodromy operator for the m-link periodic reflecting ray γ is the operator $M(k)$ on $L^2(\partial\Omega)$ defined by* (28)*.*

Proof. We define

$$M_i(k) = (-1)^{m-1}N_1(k)\Psi_{\Gamma_{i+m-1}} \cdots N_1(k)\Psi_{\Gamma_i},$$

and we show that for every i

$$r_{\partial\Omega}P_i(k) \sim \chi_{\Gamma_i} + M_i(k), \qquad (\text{microlocally near } \partial\gamma). \tag{29}$$

Since

$$\frac{1}{m}\sum_{i=1}^{m} M_i(k) = M(k), \quad \text{and} \quad \frac{1}{m}\sum_{i=1}^{m}\chi_{\Gamma_i} = I \qquad (\text{microlocally near } \partial\gamma),$$

taking averages of equations (29) over all i implies Proposition 4.4. So we only prove (29) for $i = 1$. Since $P_1(k) = \mathcal{D}\ell(k)Q_1(k)$, by the jump formula (20) and by (27), we have

$$r_{\partial\Omega}P_1(k) = (I + N(k))\sum_{j=0}^{m-1}(-1)^j\Psi_{\Gamma_{j+1}}\big[(I + N(k))\Psi_{\Gamma_j} \cdots (I + N(k))\Psi_{\Gamma_1}\big],$$

where the 0-th term of the sum is defined to be Ψ_{Γ_1}. Now we apply Proposition 4.3 to each term of the above sum and we get

$$r_{\partial\Omega} P_1(k) \sim (I + N(k)) \sum_{j=0}^{m-1} (-1)^j \Psi_{\Gamma_{j+1}} \big[N_1(k) \Psi_{\Gamma_j} \cdots N_1(k) \Psi_{\Gamma_1} \big].$$

Hence by substituting $I + N = (I + N_0) + N_1$ and collecting the terms corresponding to the same number of iterations of the billiard map β (this number is the same as the number of factors N_1 in each term) we obtain

$$1_{\partial\Omega} P_1(k) \sim \sum_{j=1}^{m-1} (-1)^{j-1} (I - \chi_{\Gamma_{j+1}}) \big[N_1(k) \Psi_{\Gamma_j} \cdots N_1(k) \Psi_{\Gamma_1} \big] + \big(\chi_{\Gamma_1} + M_1(k) \big).$$

The j-th term of the sum above is a k-FIO corresponding to β^j. We show that each of these terms is microlocally equivalent to 0 near $\partial\gamma$. Hence only χ_{Γ_1} corresponding to β^0 and $M_1(k)$ corresponding to β^m survive and the proposition follows.

Let us discuss why

$$(I - \chi_{\Gamma_{j+1}}) \big[N_1(k) \Psi_{\Gamma_j} \cdots N_1(k) \Psi_{\Gamma_1} \big] \sim 0. \qquad \text{(microlocally near } \partial\gamma)$$

This is true because

$$WF'(N_1(k) \Psi_{\Gamma_j} \cdots N_1(k) \Psi_{\Gamma_1}) \subset \{ \big(\beta^j(y,\eta), (y,\eta) \big);\ (y,\eta) \in \Gamma_1 \} \subset \Gamma_{j+1} \times \Gamma_1,$$

and because $I - \chi_{\Gamma_{j+1}}$ is micro-supported away from (v_j, η_j). Thus by passing to a smaller microlocal neighborhood of $\partial\gamma$ in $B^*\partial\Omega$, we obtain a negligible operator. □

4.2. **Microlocal parametrix for the interior Dirichlet problem in terms of $M(k)$.**

In this section we construct a microlocal parametrix for the Dirichlet resolvent of Ω near γ in terms of the monodromy operator $M(k)$. It is a microlocal version of the global formula (21) of Fredholm-Neumann.

Proposition 4.6. *Microlocally in a neighborhood of $\gamma \times \gamma \subset T^*\Omega \times T^*\Omega$, we have*

$$R^D_\Omega(k) - R_0(k) \sim -P(k)(I + M(k))^{-1} \mathcal{S}\ell(k)^t.$$

Proof. Let us look at the Schwartz kernels of the both hand sides in a microlocal neighborhood of $\gamma \times \gamma$. We show that

$$G^D_\Omega(k,x,y) - G_0(k,x,y) \sim -\big(P(k)(I + M(k))^{-1} \big) (\mathcal{S}\ell(k)^t(k,x,y)), \qquad \text{(microlocally near } \gamma\times\gamma)$$

where the operator $P(k)(I + M(k))^{-1} : C(\partial\Omega) \to C(\Omega)$ acts on the first component x of the kernel $\mathcal{S}\ell(k)^t(k,x,y)$. To prove this, since microlocal solutions are unique, it is enough to show that for all y the right side is a solution of the Dirichlet problem

$$\begin{cases} (-\Delta_x - k^2) \big(P(k)(I + M(k))^{-1} \big) (\mathcal{S}\ell(k)^t(k,x,y)) = 0, \\[2ex] r_{\partial\Omega} \big(P(k)(I + M(k))^{-1} \big) (\mathcal{S}\ell(k)^t(k,x,y)) \sim -(G^D_\Omega - G_0)(k,x,y), \quad \text{microlocally for } x \text{ near } \partial\gamma. \end{cases}$$

But this is clear because by Proposition 4.4, microlocally for $x \in \partial\Omega$ near $\partial\gamma$, we have for all $y \in \Omega$

$$r_{\partial\Omega} \big(P(k)(I + M(k))^{-1} \big) (\mathcal{S}\ell(k)^t(k,x,y)) \sim (I + M(k))(I + M(k))^{-1} \mathcal{S}\ell(k)^t(k,x,y)$$

$$\sim -(G^D_\Omega - G_0)(k,x,y).$$

We note that the operator $I + M(k)$ is invertible by Lemma 5.5. ☐

5. Trace formula and Monodromy operators

Here we assume L_γ is the only length in the support of $\hat{\rho} \in C_0^\infty$ and $\hat{\rho}(t) = 1$ near L_γ. From Proposition 4.6, we immediately have a reduction of the trace to the boundary.

Proposition **5.1.** *We have:*

$$Tr_\Omega(R^D_{\Omega,\rho}(k) - R_{0,\rho}(k)) \sim -\, Tr_{\partial\Omega}(\rho * \, k\, \mathcal{S}\ell(k)^t \mathcal{D}\ell(k) Q(k)(I + M(k))^{-1}).$$

Proof. First of all we note that the regularized trace $Tr_{\partial\Omega}(R^D_{\Omega,\rho}(k) - R_{0,\rho}(k))$ can be microlocalized to γ, i.e. if χ_γ is a microlocal cutoff around γ, then

$$Tr_{\partial\Omega}(R^D_{\Omega,\rho}(k) - R_{0,\rho}(k)) \sim Tr_{\partial\Omega}(\chi_\gamma(R^D_{\Omega,\rho}(k) - R_{0,\rho}(k))\chi_\gamma).$$

For a proof of this fact, see [Z4] §3.3. Now by Proposition 4.6, we have

$$\begin{aligned} Tr_\Omega(R^D_{\Omega,\rho}(k) - R_{0,\rho}(k)) &\sim -Tr_\Omega(\rho * \, kP(k)(I + M(k))^{-1}\mathcal{S}\ell(k)^t) \\ &\sim -Tr_{\partial\Omega}(\rho * \, k\mathcal{S}\ell(k)^t P(k)(I + M(k))^{-1}). \end{aligned}$$

The formula follows by substituting $P(k) = \mathcal{D}\ell(k)Q(k)$. ☐

This formula is useful but somewhat unwieldy. As proved in [Z3], $\mathcal{S}\ell(k)^t\mathcal{D}\ell(k) = D_0 + D_1$ where D_0 is a k-pseudodifferential operator and where D_1 quantizes β. The same analysis could be used here. But it is simpler to use the alternative in the next section.

5.1. **Interior plus exterior.** If we take the direct sum of the interior Dirichlet and exterior Neumann resolvents, then the trace formula simplifies in that we can sum up the interior and exterior $\mathcal{S}\ell(k)^t \circ \mathcal{D}\ell(k)$ operators to obtain $\frac{1}{2k}N'(k) := \frac{1}{2k}(\partial/\partial k)N(k)$.

Proposition **5.2.** *We have:*

$$Tr_{\mathbb{R}^n}(R^D_{\Omega,\rho}(k) \oplus R^N_{\Omega^c,\rho}(k) - R_{0,\rho}(k)) = -\int_{\mathbb{R}} \rho(\mu) Tr\big(N'(k-\mu)Q(k-\mu)(I + M(k-\mu))^{-1}\big)d\mu.$$

Proof. We first derive an analogue of Proposition 4.6 for the exterior Neumann problem. We then take the trace of the direct sum of the interior Dirichlet and exterior Neumann resolvents.

We construct a parametrix for the exterior Neumann problem by a modification of the method used for the interior Dirichlet problem. The discussion of the interior was motivated by the double layer representation for the interior Dirichlet Green's function. For the exterior Neumann problem we use the single layer representation of the exterior Neumann Green's function, $R^N_{\Omega^c}(k) - R_0(k) = -\mathcal{S}\ell(k)(I + N^t(k))^{-1}\mathcal{D}\ell(k)^t$, where the superscript t denotes the transpose. This formula is proved by expressing the left side as $\mathcal{S}\ell(k)\psi$ for some ψ, taking the normal derivative from the exterior and solving for ψ. We then consider

$$r_{\Omega^c}(R^N_{\Omega^c}(k) - R_0(k))r_{\Omega^c},$$

where r_X is the charateristic function of X. We observe that this operator is symmetric, i.e. equals its transpose. It follows that

$$\big(r_{\Omega^c}(R^N_{\Omega^c}(k) - R_0(k))r_{\Omega^c}\big)(x,y) = -\big(r_{\Omega^c}\mathcal{D}\ell(k)(I+N)^{-1}\mathcal{S}\ell(k)^t r_{\Omega^c}\big)(y,x).$$

Therefore, at least on the diagonal, we can use the same parametrix formula in the exterior.

We now complete the proof of Proposition 5.2 by taking the trace on $\mathbb{R}^n$ of the direct sum of the two operators,

(30)
$$\begin{cases} r_\Omega(R^D_\Omega(k) - R_0(k))r_\Omega \sim -r_\Omega P(k)(I + M(k))^{-1}\mathcal{S}\ell(k)^t r_\Omega & \text{(microlocally near } \gamma) \\ \\ r_{\Omega^c}(R^N_{\Omega^c}(k) - R_0(k))r_{\Omega^c} \sim -r_{\Omega^c} P(k)(I + M(k))^{-1}\mathcal{S}\ell(k)^t r_{\Omega^c} & \text{(microlocally near } \gamma) \end{cases}$$

In taking the trace we may cycle $\mathcal{S}\ell(k)^t r_\Omega$ to the front in the first trace and $\mathcal{S}\ell(k)^t r_{\Omega^c}$ to the front in the second trace. We then add them to get,

$$Tr_{\mathbb{R}^n}\left(R^D_\Omega(k) \oplus R^N_{\Omega^c}(k) - R_0(k)\right) \sim -Tr_{\partial\Omega}\mathcal{S}\ell(k)^t\mathcal{D}\ell(k)Q(k)(I + M(k))^{-1}, \tag{31}$$

where $\mathcal{S}\ell(k)^t \circ \mathcal{D}\ell(k)$ has the kernel

$$\int_{\mathbb{R}^n} G_0(k, x, w)\partial_{\nu_y} G_0(k, w, y)dw = \frac{1}{2k}N'(k, x, y). \tag{32}$$

For a proof of this simple fact see equation (19) of [Z5]. This indeed is why the interior Dirichlet and exterior Neumann problems were combined and explains the sense in which they are complementary. We then convolve with ρ. □

Remark **5.3.** *We notice that here the exterior trace* $Tr\big(r_{\Omega^c}(R^N_{\Omega^c,\rho}(k) - R_{0,\rho}(k))r_{\Omega^c}\big)$ *is negligible in* k *and it is added only to simplify the expression in Prop 5.1 to the more convenient expression in Prop 5.2.*

We now use Proposition 5.2 to obtain asymptotics of the trace. The next step is to expand $(I + M)^{-1}$ in a finite geometric (Neumann) series with remainder. We have

$$(I+M)^{-1} = \sum_{n=0}^{n_0}(-1)^n\, M^n + (-1)^{n_0+1}\, M^{n_0+1}(I+M)^{-1}. \tag{33}$$

The following proposition shows that, in calculating a given order of Balian-Bloch invariant $B_{\gamma,j}$, we may neglect a sufficiently high remainder of the expansion (33).

Proposition **5.4.** *Assume that* $k = \lambda + i\tau\log\lambda$. *For each order* $|k|^{-J}$ *in the trace expansion there exists* $n_0(J)$ *such that*

(*i*) $Tr\int_{\mathbb{R}}\rho(\mu)M(k-\mu)^{n_0(J)+1}(I+M(k-\mu))^{-1}N'(k-\mu)Q(k-\mu)d\mu = O(|k|^{-J-1})$,

(*ii*) $Tr_\Omega R^D_{\Omega,\rho}(k) = \sum_{n=0}^{n_0(J)}(-1)^n Tr\int_{\mathbb{R}}\rho(\mu)\, M(k-\mu)^n N'(k-\mu)Q(k-\mu)d\mu + O(|k|^{-J-1})$.

Proof. Part (*ii*) is easily proved by combining Proposition 5.2, (33) and (*i*). It remains to estimate the remainder and show (*i*). For this, we need to establish an L^2- norm estimate for the operator norm of $M(k-\mu) = M(\lambda - \mu + i\tau\log\lambda)$ for sufficiently large τ.

The proof is implied by the following norm estimate, which is analogous to Lemma 6.2 of [SZ]. Let t_0 be the constant in §3.1. We note that the monodromy operator depends on a choice of t_0 although it is not indicated in the notation.

Lemma **5.5.** *Let* $k = \lambda + i\tau\log\lambda$. *For every* $a > 0$ *there exists* $\tau > 0$ *and constants* $b, C > 0$ *such that,*

$$||M(k-\mu)||_{L^2} \leq C|k|^{-a} < \mu >^b .$$

To prove the Lemma, we observe that by Proposition 4.4,

$$||M(k-\mu)|| \leq C(|k| < \mu >)^b ||N_1(k-\mu)||^m, \tag{34}$$

For some integers C and b. We will not relabel these constants in the course of our estimates.

We recall (see (24)) that $N_1(k-\mu)$ has Schwartz kernel

$$C(k-\mu)^{n-1} e^{i(k-\mu)|x-y|} \varphi_1(|x-y|) a_1((k-\mu)|x-y|) \langle \frac{x-y}{|x-y|}, \nu_y \rangle, \tag{35}$$

where $\varphi_1(t)$ is supported in $t \geq t_0$ for some $t_0 > 0$. We estimate its norm by the Schur estimate,

(36)

$$\begin{aligned} ||N_1(k-\mu)|| &\leq C|k-\mu|^{n-1} \sup_{x \in \partial\Omega} \int_{\partial\Omega} \left| e^{i(\lambda + i\tau \log \lambda)|x-y|} \varphi_1(|x-y|) a_1((k-\mu)|x-y|) \right| dS(y) \\ &\leq C|k-\mu|^{n-1} e^{-\tau \log \lambda t_0} \sup_{x \in \partial\Omega} \int_{\partial\Omega} \left| \varphi_1(|x-y|) a_1((k-\mu)|x-y|) \right| dS(y) \\ &\leq C < \mu >^{2n} e^{-(\tau t_0 - \epsilon) \log \lambda}, \end{aligned}$$

where we estimate $|k-\mu|^{n-1} \sup_{x,y \in \partial\Omega, |x-y| \geq t_0} |a_1((k-\mu)|x-y|)| \leq C|k|^{2n} < \mu >^{2n}$.

Since we can choose any small $\epsilon > 0$ and also any large $\tau > 0$, it is clear that, for any $a > 0$, there exist ϵ and τ such that $||M(k-\mu)|| \leq C|k|^{-a} < \mu >^b$. This proves the Lemma and hence the first part of Proposition 5.4.

□

5.2. **Trace for the iterations of a bouncing ball orbit.** We now analyze the trace in part (ii) of Prop 5.4 when it is specialized to the rth iterate γ^r of a bouncing ball orbit, which has $m = 2r$ links. We observe that $Q(k-\mu)$ is a sum of terms quantizing $\beta^0, \beta^1, \ldots, \beta^{2r}$. Let us write $q_j(k-\mu)$ for the term quantizing β^j. We note that based on this notation, we have $q_{2r}(k-\mu) = M(k-\mu)$ where $M(k-\mu)$ is the monodromy operator for γ^r. It follows that

$$N'(k-\mu)\, Q(k-\mu)$$

is a sum of terms quantizing $\beta^0, \ldots, \beta^{2r+1}$. On the other hand if we use the monodromy operator for γ^r, then $(I + M(k-\mu))^{-1}$ is a sum of terms quantizing $\beta^0, \beta^{2r}, \beta^{4r}, \cdots$. Therefore only three terms of $N'(k-\mu)Q(k-\mu)(I+M(k-\mu))^{-1}$ are associated to γ^{2r} and can contribute to the trace:

(1) $-N_0'(k-\mu)\, M(k-\mu)$;
(2) $N_0'(k-\mu)\, q_{2r}(k-\mu) = N_0'(k-\mu)\, M(k-\mu)$;
(3) $N_1'(k-\mu)\, q_{2r-1}(k-\mu)$.

Here, we use that N_0' is associated to β^0 and N_1' is associated to β. We notice the terms (1) and (2) cancel and hence only the term (3) contributes to the trace. Also we notice that for the r-th iteration of a bouncing ball orbit we have only two vertices and therefore for i odd we have $\Gamma_i = \Gamma_1$ and for i even we have $\Gamma_i = \Gamma_2$. Let us denote by

$$\Gamma_+ = \Gamma_1, \quad \text{and} \quad \Gamma_- = \Gamma_2,$$

the microlocal neighborhoods corresponding to the top and bottom vertex respectively. Thus, by these notations we have

PROPOSITION 5.6. *Let $\hat{\rho} \in C_0^\infty(\mathbb{R})$ be a cut off satisfying supp $\hat{\rho} \cap Lsp(\Omega) = \{rL_\gamma\}$. Then*

(37)
$$\begin{aligned} Tr(R^D_{\Omega,\rho}(k)) &\sim -Tr \int_{\mathbb{R}} \rho(\mu) N_1'(k-\mu) q_{2r-1}(k-\mu)\, d\mu \\ &\sim \tfrac{1}{2} \sum\nolimits_{\pm} Tr \int_{\mathbb{R}} \rho(\mu)\, N_1'(k-\mu) \\ &\quad \times \overbrace{(N_1(k-\mu)\Psi_{\Gamma_\mp})(N_1(k-\mu)\Psi_{\Gamma_\pm}) \cdots (N_1(k-\mu)\Psi_{\Gamma_\mp})(N_1(k-\mu)\Psi_{\Gamma_\pm})}^{(2r-1)\ times}\, d\mu. \end{aligned}$$

We now express this trace as an explicit oscillatory integral. We consider both principal (we will define the principal terms in the next section) and non-principal terms. All terms arise as composition of $2r$ Fourier integral operators quantizing β, hence may be expressed as compositions of $2r$ oscillatory integrals. We recall that $\Psi_{\Gamma_j} = (I + N_0)^{-1}\chi_{\Gamma_j}$. Next we expand

$$(I + N_0(k))^{-1} = I + N_{-1}(k),$$

where $N_{-1}(k)$ is a (-1)st order pseudo-differential operator. If we plug $(I + N_{-1}(k))\chi_{\Gamma_\pm}$ for $\Psi_{\Gamma_\pm}$ into the expression (37), after expanding we get

COROLLARY 5.7. *Let $k = \lambda + i\tau \log \lambda$. Up to $O(|k|^{-\infty})$, the trace $Tr(R^D_{\Omega,\rho}(k))$ is a sum of $2r$ oscillatory integrals of the form*

$$I^\sigma_{2r,\rho}(k) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{(\partial\Omega)^{2r}} e^{i[\mu(t - \mathcal{L}(y_1,\ldots,y_{2r})) + k\mathcal{L}(y_1,\ldots,y_{2r})]}$$

$$A^\sigma_{2r}(k-\mu, y_1, \ldots, y_{2r})\hat{\rho}(t)\, dt\, d\mu\, dS(y_1)\cdots dS(y_r),$$

where the superscript σ; $0 \le \sigma \le 2r-1$, denotes the sum of the terms which contain σ factors of N_{-1}, and where

$$\mathcal{L}(y_1, \ldots, y_{2r}) = |y_1 - y_2| + \cdots + |y_{2r} - y_1|,$$

and $A^\sigma_{2r}(k-\mu, y_1, \ldots, y_{2r}) \in S^{-|\sigma|}_\delta((\partial\Omega)^{2r})$.

6. PRINCIPAL TERMS

The goal of this section is to identify the *principal terms*, which generate the highest derivative data, and to prove that non-principal terms contribute only lower order derivative data.

As in [Z3], we separate out a single oscillatory integral (the principal term $I^0_{2r,\rho}$) which generates all terms of the wave trace (or Balian-Bloch) expansion which contain the maximal number of derivatives of the boundary defining function per power of k (i.e. order of wave invariant).

DEFINITION 6.1. *Let γ be a 2-link periodic orbit, and let γ^r be its rth iterate. The* principal term *is the term of (37) in which $\Psi_{\Gamma_\pm}$ is replaced by $\chi_{\Gamma_\pm}$. Thus, the principal term is*

$$I^0_{2r,\rho} = -\sum_{\pm} Tr\ \rho\ *\ N_1'(\overbrace{N_1\chi_{\Gamma_\mp} N_1\chi_{\Gamma_\pm} \cdots N_1\chi_{\Gamma_\mp} N_1\chi_{\Gamma_\pm}}^{(2r-1)\ times})d\mu.$$

This oscillatory integral corresponds to $I^0_{2r,\rho}$, i.e. the one in 5.7 corresponding to $\sigma = 0$. By Corollary 5.7, the oscillatory integral $I^0_{2r,\rho}$ has the phase function $\mathcal{L}(y_1, \dots, y_{2r}) = |y_1 - y_2| + \cdots + |y_{2r} - y_1|$, where $y_p \in \partial\Omega$. We may write each y_p in graph coordinates as $(x'_p, f_\pm(x'_p))$. We will use superscripts for the $n-1$ components of x'_p, i.e. $x'_p = (x^1_p, \dots, x^{n-1}_p)$. Hence the integral is localized to $[(-\epsilon, \epsilon)^{n-1}]^{2r}$. We notice that $I^0_{2r,\rho}$ is the sum of $I^{0,+}_{2r,\rho}$ and $I^{0,-}_{2r,\rho}$, where they correspond to the $+$ and $-$ term respectively. It is clear that the phase function of $I^{0,\pm}_{2r,\rho}$ is given by

$$\mathcal{L}_\pm(x'_1, \dots, x'_{2r}) = \sum_{p=1}^{2r} \sqrt{(x'_{p+1} - x'_p)^2 + (f_{\omega_\pm(p+1)}(x'_{p+1}) - f_{\omega_\pm(p)}(x'_p))^2}.$$

Here, $w_\pm : \mathbb{Z}_{2r} \to \{\pm\}$, where $w_+(p)$ (resp. $w_-(p)$) alternates sign starting with $w_+(1) = +$ (resp. $w_-(1) = -$).

Now we have the following Theorem 6.3. First we have a definition:

Definition **6.2.** *Let γ be an m-link periodic reflecting ray, and let $\hat{\rho} \in C^\infty_0(\mathbb{R})$ be a cut off satisfying supp $\hat{\rho} \cap Lsp(\Omega) = \{rL_\gamma\}$ for some fixed $r \in \mathbb{N}$. Given an oscillatory integral $I(k)$, we write*

$$Tr R^B_{\Omega,\rho}(k) \equiv I(k) \mod \mathcal{O}(\sum_j k^{-j}(\mathcal{J}^{2j} f))$$

if

$$Tr R^B_{\Omega,\rho}(k) - I(k)$$

has a complete asymptotic expansion of the form (15), and if the coefficient of k^{-j} depends on $\le 2j$ derivatives of the defining functions f at the reflection points.

The following Theorem is the higher dimensional generalization of Theorem 4.2 of [Z3].

Theorem **6.3.** *Let $k = \lambda + i\tau \log \lambda$. Let γ be a primitive non-degenerate 2-link periodic reflecting ray, whose reflection points are points of non-zero curvature of $\partial\Omega$, and let $\hat{\rho} \in C^\infty_0(\mathbb{R})$ be a cut off satisfying supp $\hat{\rho} \cap Lsp(\Omega) = \{rL_\gamma\}$ and equals one near rL_γ for some fixed $r \in \mathbb{N}$. Orient Ω so that γ is the vertical segment $\{x' = 0\} \cap \Omega$, and so that $\partial\Omega$ is a union of two graphs over $[-\epsilon, \epsilon]^{n-1}$. Then*

(1) $Tr R^B_{\Omega,\rho}(k) \equiv I^0_{2r,\rho} \mod \mathcal{O}(\sum_j k^{-j}(j^{2j} f_\pm))$

(2) *We also have the following integral representation for $I^0_{2r,\rho}$ in the x'_p coordinates*

$$I^0_{2r,\rho} = \sum_\pm \int_{([-\epsilon,\epsilon]^{n-1})^{2r}} e^{i(k+i\tau)\mathcal{L}_\pm(x'_1,\dots,x'_{2r})} a^{pr,\pm}_{2r}(k, x'_1, x'_2, \dots, x'_{2r}) dx'_1 \cdots dx'_{2r}, \tag{38}$$

where the phase $\mathcal{L}_\pm(x'_1, \dots, x'_{2r})$ is given in (10), and where the amplitude is given by:

$$a^{pr,\pm}_{2r}(k, x'_1, \dots, x'_{2r}) = \mathcal{L}_{w_\pm}(x'_1, \dots, x'_{2r}) A^{pr,\pm}_{2r}(k, x'_1, \dots, x'_{2r}) + \frac{1}{i}\frac{\partial}{\partial k} A^{pr,\pm}_{2r}(k, x'_1, \dots, x'_{2r}).$$

Here

$$\begin{aligned} A_{2r}^{pr,\pm}(k, x_1', \ldots, x_{2r}') &= \Pi_{p=1}^{2r} \Big\{ a_1 \Big(k \sqrt{(x_p' - x_{p+1}')^2 + (f_{w_\pm(p)}(x_p') - f_{w_\pm(p+1)}(x_{p+1}'))^2} \Big) \\ &\times \frac{< x_p' - x_{p+1}' \,, \nabla_{x_p'} f_{w_\pm(p)}(x_p') > - (f_{w_\pm(p)}(x_p') - f_{w_\pm(p+1)}(x_{p+1}'))}{\sqrt{(x_p' - x_{p+1}')^2 + (f_{w_\pm(p)}(x_p') - f_{w_\pm(p+1)'}(x_{p+1}'))^2}} \Big\} \end{aligned} \tag{39}$$

where a_1 *is the Hankel amplitude in (24). Here* $w_+(p) = (-1)^{p+1}$ *and* $w_-(p) = -w_+(p)$. *Also we have identified* $x_{2r+1}' = x_1'$.

Proof. To prove the first part of Theorem it is enough to show that for a given $\sigma \geq 1$, the coefficient of k^{-j} in the stationary phase expansion of $I_{2r,\rho}^{\sigma}(k)$, has only Taylor coefficients of order at most $2j - \sigma + 1$. This is shown in §5.4 of [Z3]. The second part of Theorem follows from the proof of Proposition 3.10 of [Z3]. It is basically just eliminating the variables t and μ in the integral in Corollary (5.7) using the stationary phase lemma.

□

Theorem 6.3 is a crucial ingredient in the proof of Theorem 1. It gives explicit formula for the phase and amplitude of the principal oscillatory integrals that determine the highest order jet of Ω in each wave invariant. The notation $A_{2r}^{pr,\pm}, a_{2r}^{pr,\pm}$ refers to the amplitude of the principal terms of the $2r$th integral; these amplitudes contain terms of all orders in k and principal here does not refer to the principal symbol, i.e. the leading order term in the semi-classical expansion.

7. Stationary phase calculations of $I_{2r,\rho}^{0,\pm}$ and the wave invariants

It is easy to see that (see Proposition 4.4 of [Z3]) we have $I_{2r,\rho}^{0,+} = I_{2r,\rho}^{0,-}$. Hence it suffices to consider the + term. The oscillatory integrals $I_{2r,\rho}^{0,+}$ have the form (38) with the phase $\mathcal{L}_+$ and the amplitude $a_{2r}^{pr,+}$.

The only critical point occurs when $x_p' = 0$ for all p. We denote by $\operatorname{Hess}\mathcal{L}_\pm(0)$ the $2r(n-1) \times 2r(n-1)$ matrix with components

$$\operatorname{Hess}\mathcal{L}_\pm(0) = \left(\frac{\partial^2 \mathcal{L}_\pm}{\partial x_p^i \partial x_q^j} \right), \quad i, j = 1, \ldots, n-1; \; p, q = 1, \ldots, 2r. \tag{40}$$

It is the Hessian of $\mathcal{L}_\pm$ at its critical point $(x_1', \ldots, x_{2r}') = 0$ in Cartesian coordinates.

We denote by $\mathcal{H}_+$ the inverse Hessian operator in the variables $(x_1', \ldots, x_{2r}')$ at this critical point. That is $\mathcal{H}_+ = \langle Hess(\mathcal{L}_+)^{-1}(0) D, D \rangle$, where D is short for $(\frac{\partial}{\partial x_1^1}, \cdots \frac{\partial}{\partial x_{2r}^{n-1}})$. More precisely,

$$\mathcal{H}_+ = \sum_{p,q=1}^{2r} \sum_{i,j=1}^{n-1} h^{(i,j),(p,q)} \left(\frac{\partial^2}{\partial x_p^i \partial x_q^j} \right). \tag{41}$$

Before we apply the Stationary Phase Lemma, in two subsections we state some properties of the inverse Hessian matrix of $\mathcal{L}_+$, and also some properties of the phase function $\mathcal{L}_+$ and principal amplitude $a_{2r}^{pr,+}$ which may be derived directly from the formula in Theorem 6.3.

7.1. **Properties of Hess $(\mathcal{L}_+)^{-1}$.** Let $\{\nu_{j,\pm}\}_{j=1}^{n-1}$ denote the eigenvalues of the second fundamental form of $\partial\Omega$ at the endpoints of the bouncing ball orbit. Without loss of generality we can assume

$$\nu_{j,\pm} = D^2_{x^j} f_\pm(0), \qquad j = 1, ..., n-1. \tag{42}$$

This is because by an orthogonal change of variable (i.e. an isometry of the plane) we can make Hess $f_\pm$ a diagonal matrix. Of course when all the symmetry assumptions are satisfied, then Hess $f_\pm$ is automatically diagonal.

The following generalizes Proposition 2.2 of [Z3].

Proposition **7.1.** *Put $a_{j,\pm} = -2(1 \pm L\nu_{j,\pm})$, and let $A_\pm = Diag(a_{j,\pm})$ be the $(n-1)\times(n-1)$ diagonal matrix with the diagonal entries $a_{j,\pm}$ Then the Hessian H_{2r} of $\mathcal{L}_+$ at $x' = 0$ has the form:*

$$H_{2r} = \frac{-1}{L}\left\{\begin{array}{ccccc} A_+ & I & 0 & \dots & I \\ I & A_- & I & \dots & 0 \\ 0 & I & A_+ & I & 0 \\ \dots & \dots & \dots & \dots & \dots \\ I & 0 & 0 & \dots & A_- \end{array}\right\},$$

where there are $2r \times 2r$ blocks and each block is of size $(n-1)\times(n-1)$.

Proof. There are $2r$ sets of variables x'_p and therefore there are $2r \times 2r$ blocks and the (p,q)-th block is given by $D^2_{x'_p x'_q}\mathcal{L}_+(0)$. We have:

(43)
$$\nabla_{x'_p}\mathcal{L}_\pm = \frac{(x'_p - x'_{p+1}) + (f_{w_\pm(p)}(x'_p) - f_{w_\pm(p+1)}(x'_{p+1}))\nabla_{x'_p} f_{w_\pm(p)}(x'_p)}{\sqrt{(x'_p - x'_{p+1})^2 + (f_{w_\pm(p)}(x'_p) - f_{w_\pm(p+1)}(x'_{p+1}))^2}} - \frac{(x'_{p-1} - x'_p) + (f_{w_\pm(p-1)}(x'_{p-1}) - f_{w_\pm(p)}(x'_p))\nabla_{x'_p} f_{w_\pm(p)}(x'_p)}{\sqrt{(x'_p - x'_{p-1})^2 + (f_{w_\pm(p)}(x'_p) - f_{w_\pm(p-1)}(x'_{p-1}))^2}}.$$

A simple calculation (using (43)) shows that all the blocks $D^2_{x'_p x'_q}\mathcal{L}_+(0)$ are zero except the ones with $p = q, p = q+1$ and $q = p+1$. From (43) we obtain

$$\begin{cases} D^2_{x'_p x'_p}\mathcal{L}_+(0) = 2\big(\frac{1}{L}\, I + w_+(p)\text{Hess}\, f_{w_+(p)}(0)\big) = \frac{-1}{L}\, A_{w_+(p)}, \\ \\ D^2_{x'_p x'_{p+1}}\mathcal{L}_+(0) = \frac{-1}{L}\, I. \end{cases}$$

□

In the elliptic case, $\det H_{2r}$ is a polynomial in $\cos\alpha_j/2$ (in $\cosh\alpha_j/2$ in the hyperbolic case) of degree $2r(n-1)$. Here, in the elliptic case, $\{e^{\pm i\alpha_1}, ... e^{\pm i\alpha_{n-1}}\}$ are the eigenvalues of the Poincare map $\mathcal{P}_\gamma$.

Proposition **7.2.** *We have*

$$\det H_{2r} = L^{2r(1-n)} \prod_{j=1}^{n-1} (2 - 2\cos r\alpha_j).$$

We will use Proposition 7.3 in the following subsection to prove Proposition 7.2.

7.1.1. *Poincaré map and Hessian of the length functional.* The linear Poincaré map P_γ of γ is the derivative at $\gamma(0)$ of the first return map to a transversal to Φ^t at $\gamma(0)$. By a non-degenerate periodic reflecting ray γ, we mean one whose linear Poincaré map P_γ has no eigenvalue equal to one. For the definitions and background, we refer to [PS, KT].

There is an important relation between the spectrum of the Poincaré map P_γ of a periodic m-link reflecting ray and the Hessian H_m of the length functional at the corresponding critical point of $L : \mathbf{T}^m \to \mathbb{R}$. For the following, see [KT] (Theorem 3).

PROPOSITION **7.3.** *Let γ be a periodic m-link reflecting ray in plane domain Ω. Then we have:*

$$\det(I - P_\gamma) = -\det(H_m)\cdot(b_1\cdots b_m)^{-1},$$

where $b_p = \frac{\partial^2|q(\varphi_{p+1} - q(\varphi_p)|}{\partial\varphi_p\partial\varphi_{p+1}}$.

Proposition 7.3 is stated only for the plane domains. One can probably prove it for higher dimensions, but the formulae for the plane domains is enough for us to prove Prop.7.2

Proof of Prop.7.2. Let us first assume $n = 2$. Let $\lambda_r, \lambda_r^{-1}$ be the eigenvalues of P_{γ^r}, so that $\det(I - P_{\gamma^r}) = 2 - (\lambda_r + \lambda_r^{-1})$. Since all the coefficients b_j are equal to $1/L$, from Prop.7.3 it follows that

$$\det(I - P_{\gamma^r}) = L^{2r}\det H_{2r}; \quad (\gamma \;\; 2-\text{link.}) \tag{44}$$

Now, if the eigenvalues of P_γ are $\{e^{\pm i\alpha}\}$ (say, in the elliptic case) then those of P_{γ^r} are $\{e^{\pm ir\alpha}\}$, hence the left side of (44) equals $2 - 2\cos r\alpha$. Now assume $n \geq 2$ and assume $\{e^{\pm i\alpha_1}, ...e^{\pm i\alpha_{n-1}}\}$ are the eigenvalues of P_γ. We just showed that if we define

$$H_{j,2r} = \frac{-1}{L}\left\{\begin{array}{ccccc} a_{j,+} & 1 & 0 & \dots & 1 \\ 1 & a_{j,-} & 1 & \dots & 0 \\ 0 & 1 & a_{j,+} & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & a_{j,-} \end{array}\right\}_{2r\times 2r},$$

then $\det H_{j,2r} = L^{-2r}(2 - 2\cos(r\alpha_j))$. Now we notice because all the blocks of the matrix H_{2r} are diagonal matrices, therefore they commute and we can write

$$\det H_{2r} = \det\big(\mathrm{Diag}(\det H_{1,2r}, \dots, \det H_{n-1,2r})\big) = L^{2r(1-n)}\prod_{j=1}^{n-1}(2 - 2\cos r\alpha_j).$$

We now consider the inverse Hessian $\mathcal{H}_+ = H_{2r}^{-1}$, which will be important in the calculation of wave invariants. We denote its matrix elements by $h_+^{ij,pq}$ which corresponds to the (i,j)-th entry of the (p,q)-th block of the matrix $\mathcal{H}_+$. We also denote by $\mathcal{H}_-$ the matrix in which the roles of A_+, A_- are interchanged; it is the Hessian of $\mathcal{L}_-$. We also notice since H_{2r} is a block matrix with each block a diagonal matrix so is its inverse $\mathcal{H}_+$. Hence the only non-zero entries of the inverse Hessian $\mathcal{H}_+$ are of the form $h_+^{ii,pq}$.

PROPOSITION **7.4.** *The diagonal matrix elements $h_+^{ii,pp}$ are constant when the parity of p is fixed, and for every $1 \leq i \leq n-1$ we have:*

$$p \ \ odd \implies h_\pm^{ii,pp} = h_\pm^{ii,11}, \qquad p \ \ even \implies h_\pm^{ii,pp} = h_\pm^{ii,22}$$

$$h_+^{ii,11} = h_-^{ii,22}, \quad h_+^{ii,22} = h_-^{ii,11} \qquad .$$

Proof. It is enough to show this for $n = 2$. This is because H_{2r} is a block matrix with commuting blocks and each block is diagonal. In fact based on our definition in (7.1.1) we have

$$h^{ii,pq} = \big(H_{2r}\big)^{ii,pq} = \big(H_{i,2r}\big)^{pq}. \tag{45}$$

Hence it is enough to prove the proposition for the entries of the inverse of the $2r \times 2r$ matrix $H_{i,2r}$. This reduces the problem to $n - 1 = 1$.

Now let us introduce the cyclic shift operator on $\mathbb{R}^{2r}$ given by $Pe_j = e_{j+1}$, where $\{e_j\}$ is the standard basis, and where $Pe_{2r} = e_1$. It is then easy to check that $P\mathcal{H}_+P^{-1} = \mathcal{H}_-$, hence that $P\mathcal{H}_+^{-1}P^{-1} = \mathcal{H}_-^{-1}$. Since P is unitary, this says

$$h_-^{pq} = \langle \mathcal{H}_- e_p, e_q \rangle = \langle P\mathcal{H}_+^{-1}P^{-1}e_p, e_q \rangle = \langle \mathcal{H}_+^{-1}P^{-1}e_p, P^{-1}e_q \rangle = h_+^{p-1,q-1}.$$

It follows that the matrix $\mathcal{H}_\pm$ is invariant under even powers of the shift operator, which shifts the indices $p \to p + 2k$ $(k = 1, \dots, r)$. Hence, diagonal matrix elements of like parity are equal.

□

7.1.2. *Inverse Hessian at $(\frac{\mathbb{Z}}{2\mathbb{Z}})^n$-symmetric bouncing ball orbits.* We first observe that in the case of $(\frac{\mathbb{Z}}{2\mathbb{Z}})^n$ symmetric domains, the $(2r)(n-1) \times (2r)(n-1)$ Hessian of Proposition 7.1 simplifies to:

$$H_{2r} = \frac{-1}{L} \left\{ \begin{array}{cccccc} A & I & 0 & \dots & I \\ I & A & I & \dots & 0 \\ 0 & I & A & I & 0 \\ 0 & 0 & I & A & I \dots \\ \dots & \dots & \dots & \dots & \dots \\ I & 0 & 0 & \dots & A \end{array} \right\} \tag{46}$$

which is a $2r \times 2r$ block matrix, each block of size $(n-1) \times (n-1)$. Here $A = Diag(a_j)$ and I is the rank $n-1$ identity matrix. The diagonal entries a_j are given by

(47)

$a_j = 2\cos\alpha_j/2 = -2(1+L\nu_j)$ (elliptic case), $\quad a_j = 2\cosh\alpha_j/2 = -2(1+L\nu_j)$ (hyperbolic case).

We can express the inverse Hessian matrix elements $h_{2r}^{ij,pq}$ in terms of Chebychev polynomials T_m, resp. U_m, of the first, resp. second, kind. The Chebychev polynomials are defined by:

$$T_m(\cos\theta) = \cos m\theta, \qquad U_m(\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}.$$

PROPOSITION **7.5.** [Z3] *Suppose that H_{2r} is given by (46). Then the matrix elements of H_{2r}^{-1} are given by*

$$h_{2r}^{ii,pq} = \tfrac{1}{2[1-T_{2r}(-a_i/2)]}[U_{2r-q+p-1}(-a_i/2) + U_{q-p-1}(-a_i/2)], \quad (1 \le p \le q \le 2r;\ 1 \le i \le n-1)$$

$$h_{2r}^{ij,pq} = 0, \qquad i \neq j.$$

Proof. This formula was proved in [Z3] for the case $n = 2$. For the general case $n \geq 2$, we just use the equation (45) and reduce it to $n = 2$. □

We note that $h^{ij,pq} = h^{ij,qp}$ so this formula determines all of the matrix elements. It follows in the elliptic case that

$$h_{2r}^{ii,pq} = \begin{cases} \frac{(-1)^{p-q}}{2(1-\cos r\alpha_i)}\left(\frac{\sin(2r-q+p)\alpha_i/2}{\sin\alpha_i/2} + \frac{\sin(q-p)\alpha_i/2}{\sin\alpha_i/2}\right) & (1 \le p \le q \le 2r) \\ \frac{(-1)^{p-q}}{2(1-\cos r\alpha_i)}\left(\frac{\sin(2r-p+q)\alpha_i/2}{\sin\alpha_i/2} + \frac{\sin(p-q)\alpha_i/2}{\sin\alpha_i/2}\right) & (1 \le q \le p \le 2r) \end{cases} \tag{48}$$

There are obvious analogues in the hyperbolic and mixed cases.

The case of interest to us is

$$h_{2r}^{ii,11} = \frac{1}{2(1-\cos r\alpha_i)}\frac{\sin r\alpha_i}{\sin\alpha_i/2} = \frac{1}{\sin\alpha_i/2}\cot\frac{r\alpha_i}{2}. \tag{49}$$

7.2. **Properties of the phase function $\mathcal{L}_+$ and the amplitude $a_{2r}^{pr,+}$.** Since $\mathcal{L}_+$ and $a_{2r}^{pr,+}$ are functions of $2r(n-1)$ variables x_p^j where $1 \le p \le 2r$ and $1 \le j \le n-1$, to simplify our notations we denote:

Let $[\gamma] = (\gamma_p^j)$, $1 \le p \le 2r$, $1 \le j \le n-1$ be a $2r \times (n-1)$ matrix of indices. We let $m = |[\gamma]| = \sum_{p,j}\gamma_p^j$. Then we define

$$D_{[\gamma]}^m = \frac{\partial^m}{(\partial x_1^1)^{\gamma_1^1}\ldots(\partial x_{2r}^{n-1})^{\gamma_{2r}^{n-1}}}.$$

We will use $\vec{\gamma}_p$ for the p-th row of $[\gamma]$, and sometimes if $\vec{\gamma}_p, \vec{\gamma}_q, \ldots$ are the only non-zero rows of $[\gamma]$ we write $D^m_{\vec{\gamma}_p,\vec{\gamma}_q,\ldots}$ for $D^m_{[\gamma]}$ to emphasize that $\vec{\gamma}_p, \vec{\gamma}_q, \ldots$ are the only non-zero rows of $[\gamma]$. The calculation of the highest derivative terms of the Balian-Bloch wave invariants uses only the following properties of the phase and principal amplitude which may be derived directly from the formulae in Theorem 6.3.

The following Lemma is the higher dimensional generalization of Lemma 4.5 of [Z3]. It is proved in the same way, and the proof is therefore omitted.

LEMMA **7.6.** *The phase and principal amplitude of the principal oscillatory integrals $I_{2r,\rho}^{0,\pm}$ have the following properties:*

(i) In its dependence on the boundary defining functions $f_\pm$, *the amplitude* $a_{2r}^{pr,+}$ *has the form* $\alpha_r(k, x', f_\pm, f'_\pm)$.

(ii) As above, in its dependence on x'

$a_{2r}^{pr,+}(k, x'_1, \dots, x'_{2r}) = \mathcal{L}_+(x'_1, \dots, x'_{2r}) A_{+,r}^{pr}(k, x'_1, \dots, x'_{2r}) + \frac{1}{i}\frac{\partial}{\partial k} A_{2r}^{pr,+}(k, x'_1, \dots, x'_{2r})$, *where* $A_{2r}^{pr,+}(k, x'_1, \dots, x'_{2r}) \;=\; \Pi_{p=1}^{2r} A_p(x'_p, x'_{p+1}) \quad (2r+1 \equiv 1)$

(iii)At the critical point, the principal amplitude has the asymptotics

$a_{2r}^{pr,+}(k, 0) \sim (2rL) L^{-(n-1)r} \mathcal{A}(r) + O(k^{-1})$, *where* $\mathcal{A}(r)$ *depends only on* r *and not on* Ω;

(iiia) $\frac{a_{2r}^{pr,+}(k,0) e^{i(k+i\tau)\mathcal{L}_+(0) + i\pi/4 sgn Hess \mathcal{L}_+(0)}}{\sqrt{\det Hess \mathcal{L}_+}} \sim (2rL)\; \mathcal{A}(r)\; \mathcal{D}_{D,\gamma^r}(k)(1 + O(k^{-1}))\; (cf.15)$;

(iv) $\nabla a_{2r}^{pr,+}(k, x'_1, \dots, x'_{2r})|_{x'=0} = 0$.

(v.a) $D_{\vec{\gamma}_p}^{2j+1} \mathcal{L}_+|_{x'=0} \equiv 2w_+(p) D_{\vec{\gamma}_p}^{2j+1} f_{w_+(p)}(0) \quad mod \;\; R_{2r}(\mathcal{J}^{2j} f_+(0), \mathcal{J}^{2j} f_-(0))$,

(v.b) $D_{\vec{\gamma}_p}^{2j+2} \mathcal{L}_+|_{x=0} \equiv 2w_+(p) D_{\vec{\gamma}_p}^{2j+2} f_{w_+(p)}(0) \; mod \; R_{2r}(\mathcal{J}^{2j} f_+(0), \mathcal{J}^{2j} f_-(0))$,

(v.c)If $[\gamma]$ *has more than one non-zero row, say* $\vec{\gamma}_p, \vec{\gamma}_q, \dots$, *then*

$$D_{\vec{\gamma}_p, \vec{\gamma}_q, \dots}^{2j+1} \mathcal{L}_+(0) \equiv 0 \quad mod \;\; R_{2r}(\mathcal{J}^{2j} f_+(0), \mathcal{J}^{2j} f_-(0)),$$

and

$$D_{\vec{\gamma}_p, \vec{\gamma}_q, \dots}^{2j+2} \mathcal{L}_+(0) \equiv 0 \quad mod \;\; R_{2r}(\mathcal{J}^{2j} f_+(0), \mathcal{J}^{2j} f_-(0)),$$

where $\equiv$ *in general means equality modulo lower order derivatives of* f.

7.3. **Stationary phase diagrammatics.** We briefly review the stationary phase expansion from the diagrammatic point of view. For more details we refer to [A, E, Z3].

The stationary phase expansion gives an asymptotic expansion for an oscillatory integral

$$Z(k) = \int_{\mathbb{R}^n} a(x) e^{ikS(x)} dx$$

where $a \in C_0^\infty(\mathbb{R}^n)$ and where S has a unique non-degenerate critical point in supp(a) at $x = 0$. Let us write H for the Hessian of S at 0. The stationary phase expansion is:

$$Z(k) = (\frac{2\pi}{k})^{n/2} \frac{e^{i\pi sgn(H)/4}}{\sqrt{|det H|}} e^{ikS(0)} Z_A(k),$$

where

$$Z_A(k) = \sum_{j=0}^{\infty} k^{-j} \sum_{(\Gamma,\ell) \in \mathcal{G}_{V,I},\, I-V=j} \frac{I_\ell(\Gamma)}{S(\Gamma)},$$

where $\mathcal{G}_{V,I}$ consists of labelled graphs (Γ, ℓ) with V closed vertices of valency ≥ 3 (each corresponding to the phase), with one open vertex (corresponding to the amplitude), and with I edges. The function ℓ 'labels' each end of each edge of Γ with an index $j \in \{1, \dots, n\}$.

Above, $S(\Gamma)$ denotes the order of the automorphism group of Γ, and $I_\ell(\Gamma)$ denotes the 'Feynman amplitude' associated to (Γ, ℓ). By definition, $I_\ell(\Gamma)$ is obtained by the following rule: To each edge with end labels j, k one assigns a factor of ih^{jk} where $H^{-1} = (h^{jk})$. To each closed vertex one assigns a factor of $i\frac{\partial^\nu S(0)}{\partial x^{i_1} \cdots \partial x^{i_\nu}}$ where ν is the valency of the vertex and $i_1 \dots, i_\nu$ at the index labels of the edge ends incident on the vertex. To the open vertex, one assigns the factor $\frac{\partial^\nu a(0)}{\partial x^{i_1} \dots \partial x^{i_\nu}}$, where ν is its valence. Then $I_\ell(\Gamma)$ is the product of all these factors. To the empty graph one assigns the amplitude 1. In summing over (Γ, ℓ) with a fixed graph Γ, one sums the product of all the factors as the indices run over $\{1, \dots, n\}$.

We note that the power of k in a given term with V vertices and I edges equals $k^{-\chi_{\Gamma'}}$, where $\chi_{\Gamma'} = V - I$ equals the Euler characteristic of the graph Γ' defined to be Γ minus the open vertex. We note that there are only finitely many graphs for each χ because the valency condition forces $I \geq 3/2V$. Thus, $V \leq 2j, I \leq 3j$.

7.4. **The stationary phase calculations of $I^{0,+}_{2r,\rho}$: The data $f^{(2j+2)}_{\pm}(0)$.** In this section we will repeatedly use different parts of Lemma 7.6 without quoting them.

We first claim that in the stationary phase expansion of $I^{0,+}_{2r,\rho}$, the data $f^{(2j+2)}_{\pm}(0)$ appears first in the k^{-j} term . This is because any labelled graph (Γ, ℓ) for which $I_\ell(\Gamma)$ contains the factor $f^{(2j+2)}_{\pm}(0)$ must have a closed vertex of valency $\geq 2j+2$, or the open vertex must have valency $\geq 2j+1$. The minimal absolute Euler characteristic $|\chi(\Gamma')|$ in the first case is j. Since the Euler characteristic is calculated after the open vertex is removed, the minimal absolute Euler characteristic in the second case is $j+1$ (there must be at least $j+1$ edges.) Hence the latter graphs do not have minimal absolute Euler characteristic. More precisely, we have:

Proposition **7.7.** *In the stationary phase expansion of $I^{0,+}_{2r,\rho}$, the only labelled graph (Γ, ℓ) with $\chi(\Gamma') = V - I = -j$ and $I_\ell(\Gamma)$ containing $f^{(2j+2)}_{\pm}(0)$ is given by:*

- $\Gamma_{1,j+1} \in \mathcal{G}_{1,j+1}$ *(i.e. $V = 1, I = j+1$). The graph $\Gamma_{1,j+1}$ has no open vertex, one closed vertex and $j+1$ loops at the closed vertex.*
- *The only labels producing the desired data are those ℓ_p, with $1 \leq p \leq 2r$ fixed, which labels all endpoints of all edges as (i, p) where $1 \leq i \leq n-1$. (Notice the label (i, p) corresponds to the variable x^i_p.)*

In addition, the sum of the Feynman amplitudes corresponding to the labelled graphs $(\Gamma_{1,j+1}, \ell_p)$ above, for a fixed p, is

$$\sum_{\ell_p} I_{\ell_p}(\Gamma_{1,j+1}) \equiv (4rL)L^{-(n-1)r}\mathcal{A}(r)\{i^{j+2} \sum_{|\vec{\gamma}_p| = j+1} \frac{(j+1)!}{\vec{\gamma}_p!}(h_+^{1\vec{1},pp})^{\vec{\gamma}_p}\, w_+(p) D^{2j+2}_{2\vec{\gamma}} f_{w_+(p)}(0)$$

where we neglect terms with $\leq 2j+1$ derivatives.

Proof. We argued diagrammatically that the power k^{-j} is the greatest power of k in which $f^{(2j+2)}_{\pm}(0)$ appears. We also showed that a labelled graph with Euler characteristic $-j$ which produces $f^{(2j+2)}_{\pm}(0)$ must have a closed vertex of valency $\geq 2j+2$. Now it is clear that such graph must have only one closed vertex and $j+1$ loops. This proves the first part of the proposition. The second part follows easily from Proposition 7.6.

Now let us determine $\sum_{\ell_p} I_{\ell_p}(\Gamma_{1,j+1})$ for the labelled graphs $(\Gamma_{1,j+1}, \ell_p)$ above. We have

(50)
$$\sum_{\ell_p} I_{\ell_p}(\Gamma_{1,j+1}) \equiv (2rL)L^{-(n-1)r}\mathcal{A}(r)\{i^{j+2} \sum_{\gamma_p^1+\ldots+\gamma_p^{n-1}=j+1} \frac{(j+1)!}{\gamma_p^1!\ldots\gamma_p^{n-1}!}(h_+^{11,pp})^{\gamma_p^1}\ldots(h_+^{(n-1)(n-1),pp})^{\gamma_p^{n-1}} D^{2j+2}_{\vec{\gamma_p}}\mathcal{L}_+(0).$$

So by Proposition 7.6 and using short-hand notations we get

$$\sum_{\ell_p} I_{\ell_p}(\Gamma_{1,j+1}) \equiv (4rL)L^{-(n-1)r}\mathcal{A}(r)\{i^{j+2} \sum_{|\vec{\gamma}_p|=j+1} \frac{(j+1)!}{\vec{\gamma}_p!}(h_+^{\vec{11},pp})^{\vec{\gamma}_p}\, w_+(p)D^{2j+2}_{2\vec{\gamma}} f_{w_+(p)}(0)$$

□

7.5. **Proof of Theorem 2.** Now we are ready to prove Theorem 2. The discussion above shows that modulus derivatives of order $\leq 2j+1$

$$B_{\gamma^r,j} \equiv (2rL)^{-1}L^{(n-1)r}\sum_{p=1}^{2r} \frac{\sum_{\ell_p} I_{\ell_p}(\Gamma_{1,j+1})}{S(\Gamma_{1,j+1})}.$$

We notice that $S(\Gamma_{1,j+1}) = |Aut(\Gamma_{1,j+1})| = 2^{j+1}(j+1)!$. We then break up the sums over p of even/odd parity and use Proposition 7.4 to replace the odd parity Hessian elements by h_+^{11} and the even ones by h_+^{22}. Taking into account that $w_+(p) = 1(-1)$ if p is even (odd), we conclude that (by the formula in Proposition 7.7)

$$B_{\gamma^r,j} = \frac{B_{\gamma^r,0}}{(2i)^{j+1}} \sum_{|\gamma|=j+1} \frac{r}{\vec{\gamma}!}\big\{(\overrightarrow{h^{11}_{+,2r}})^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f_+(0) - (\overrightarrow{h^{11}_{-,2r}})^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f_-(0)\big\}.$$

So far we have proved all parts of Theorem 2 except the last part which finds a formula for the wave invariants in the case of symmetries.

7.5.1. *Balian-Bloch invariants at bouncing ball orbits of* $(\frac{\mathbb{Z}}{2\mathbb{Z}})^n$ *symmetric domains.* Now if we assume the $(\frac{\mathbb{Z}}{2\mathbb{Z}})^n$ symmetry assumptions, namely $f_+ = f = -f_-$ and f being even in all variables, then using (49) the formula above simplifies to

(51)
$$B_{\gamma^r,j} = \tfrac{B_{\gamma^r,0}}{(2i)^{j+1}} \textstyle\sum_{|\vec{\gamma}|=j+1} \tfrac{r}{\vec{\gamma}!}\left(\tfrac{1}{\sin\frac{\vec{\alpha}}{2}}\cot\tfrac{r\vec{\alpha}}{2}\right)^{\vec{\gamma}} D^{2j+2}_{2\vec{\gamma}} f(0)$$
$$+\{\text{polynomial of Taylor coefficients of order} \leq 2j\}.$$

This finishes the proof of Theorem 2. Q.E.D.

7.6. **Recovering the Taylor Coefficients and the Proof of Theorem 1.** First of all we prove the following lemma

LEMMA **7.8.** *If* $\{\alpha_1, \ldots \alpha_{n-1}\}$ *are linearly independent over* $\mathbb{Q}$ *then the functions*

$$\left(\cot\frac{r\vec{\alpha}}{2}\right)^{\vec{\gamma}}$$

are linearly independent as functions of $r \in \mathbb{N}$.

Proof. Suppose that there exist coefficients $c_{\vec{\gamma}}$ such that

$$\sum_{\vec{\gamma}} c_{\vec{\gamma}} \left(\cot \frac{r\vec{\alpha}}{2}\right)^{\vec{\gamma}} = 0, \quad \forall r \in \mathbb{N}.$$

Consider the function

$$\psi(z_1, \ldots, z_{n-1}) := \sum_{\vec{\gamma}} c_{\vec{\gamma}} \left(\cot \vec{z}\right)^{\vec{\gamma}}.$$

This function is meromorphic and periodic of period 2π in each variable z_j, so it may be regarded as a meromorphic function on $(\mathbb{C}/\mathbb{Z})^{n-1}$. It vanishes when $z_j = r\alpha_j/2$ modulo 2π for all $r = 1, 2, 3, \ldots$. But such points are dense in the real submanifold $(\mathbb{R}/\mathbb{Z})^{n-1}$ and hence the function vanishes identically on $(\mathbb{C}/\mathbb{Z})^{n-1}$. This is a contradiction since the functions $\Pi_{j=1}^{n-1} w_j^{\gamma_j}$ are independent functions and by the change of variables $w_j = \cot z_j$ the functions $\Pi_{j=1}^{n-1} (\cot z_j)^{\gamma_j}$ must also be independent. $\square$

Now assume $\Omega \subset \mathbb{R}^n$ is a domain in the class $\mathcal{D}_L$ defined in (3). Take a non-degenerate bounding ball orbit γ of length $2L$ which satisfies all the properties listed in (3). We would like to use mathematical induction and recover the Taylor coefficients of the function f where f and $-f$ are the local defining functions of $\partial\Omega$ near the top and bottom of the bouncing ball orbit respectively. First, it is possible to recover all the α_j, $1 \le j \le n-1$, under a permutation [Fr]. This is because $|\det(I - P_{\gamma^r})|$ is a spectral invariant (the 0-th wave invariant). But we know that $|\det(I - P_{\gamma^r})| = \prod_{j=1}^{n-1}(2 - 2\cos(r\alpha_j))$. Hence $\prod_{j=1}^{n-1}(\sin^2(r\alpha_j/2))$ is a spectral invariant for all $r \in \mathbb{N}$. It is easy to see that this condition determines α_j under a permutation. We fix this permutation and we argue inductively to recover all the Taylor coefficients. Since f is even in all the variables, the odd order Taylor coefficients are zero. Now assume $D_{2\vec{\gamma}}^{2|\vec{\gamma}|} f(0)$ are given for all $|\vec{\gamma}| \le j$. Hence the remainder polynomial term of (51) is given. Now by the above lemma, since all the functions $\left(\cot \frac{r\vec{\alpha}}{2}\right)^{\vec{\gamma}}$ are linearly independent, we can recover the Taylor coefficients $D_{2\vec{\gamma}}^{2j+2} f(0)$. This concludes the proof of Theorem 1.

The analogous arguments will follow in the hyperbolic or mixed hyperbolic-elliptic cases.

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Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: `zelditch@math.jhu.edu, hhezari@math.jhu.edu`